

SEIDEL-SMITH COHOMOLOGY FOR TANGLES

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ABSTRACT. We generalize the “symplectic Khovanov cohomology” of Seidel and Smith [17] to tangles using the notion of symplectic valued topological field theory introduced by Wehrheim and Woodward [19].

1. INTRODUCTION

In the year 2000, Mikhail Khovanov [9] introduced a doubly graded homology theory of links which categorifies the Jones polynomial, i.e. whose graded Euler characteristic equals the Jones polynomial. He later generalized this cohomology theory to even tangles (i.e. tangles with even number of initial and endpoints)[10]. In that paper he introduced a family of rings $\{H^n\}_{n \in \mathbb{N}}$ and to each (m, n) -tangle he assigned a complex of (H^m, H^n) bimodules. By means of tensor product one obtains, for each tangle, a functor on the category of such complexes modulo homotopy. In 2003, P. Seidel and I. Smith defined a singly graded link cohomology based on symplectic geometry which they called “symplectic Khovanov homology”. They conjectured that this invariant equaled Khovanov homology after the collapse of the bigrading. They defined a family $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$ of symplectic manifolds and to each braid $\beta \in Br_{2m}$ they assigned a symplectomorphism h_β of \mathcal{Y}_m . (See section 2.) The Seidel-Smith invariant of a link K is the Floer cohomology of L and $h_\beta(L)$ where L is a specific Lagrangian submanifold of \mathcal{Y}_m and β is any braid representation of K . They prove that this is independent of the choice of the braid representation β . Later C. Manolescu [12] gave a more explicit description of the invariant and equipped the chain complex with a second grading, showing that the Euler characteristic of this chain complex equals the Jones polynomial. However it is not known if this grading descends to a grading on cohomology.

In this paper we construct a generalization of the Seidel-Smith invariant to even tangles. To any elementary (i, j) -tangle T we assign a Lagrangian correspondence L_T between \mathcal{Y}_i and \mathcal{Y}_j . If T is a braid, we assign to it the graph of the symplectomorphism h_T defined by Seidel and Smith. The remaining elementary tangles are caps and cups. (See Figure 5.) To a $(m, m+2)$ cap we assign a vanishing cycle over the diagonal $\Delta \subset \mathcal{Y}_m \times \mathcal{Y}_m$. The Lagrangian assigned to a cup is the transpose of this vanishing cycle. See section 4.2. Now any given (m, n) -tangle \mathcal{T}

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can be decomposed into a composition of elementary ones

$$\mathcal{T} = T_k T_{k-1} \cdots T_1.$$

To \mathcal{T} we assign the *generalized* Lagrangian correspondence

$$\Phi(\mathcal{T}) = (L_{T_k}, L_{T_{k-1}}, \cdots, L_{T_1})$$

between \mathcal{Y}_m and \mathcal{Y}_n . We then prove the following.

Theorem 4.2.8. *Up to isomorphism of generalized correspondences, $\Phi(\mathcal{T})$ is independent of the decomposition of \mathcal{T} into elementary tangles.*

This way we obtain two invariants for each (m, n) -tangle T ; The first one is a functor $\Phi_T^\#$ from the generalized Fukaya category of \mathcal{Y}_m to that of \mathcal{Y}_n . The category used here is an enlargement of the Fukaya category of a Stein manifold to include a special class of noncompact Lagrangians. The second one is a graded abelian group, denoted $\mathcal{K}_{\text{symp}}(T)$, which is, roughly, the Floer cohomology of $\Phi(T)$. For this second invariant to be well-defined we first have to deal with the compactness of the involved moduli spaces. The reason is that the Lagrangians assigned to caps and cups are not compact. We prove compactness using standard (but not very well-known) arguments on Lagrangians in manifolds with contact type boundary. In sections 3.1 and 3.3 we put together necessary tools for construction of Floer homology of noncompact Lagrangians in Stein manifolds. From these plus Theorem 4.2.8 and the Functoriality Theorem of [19] we get the following.

Theorem 4.3.3. *$\mathcal{K}_{\text{symp}}(T)$ is well-defined and is independent of the decomposition of T into elementary tangles.*

An algebraic-geometric equivalent of Khovanov homology has been developed by S. Cautis and J. Kamnitzer [1]. In that paper they assign to each elementary (k, l) -tangle a Fourier-Mukai kernel between specific algebraic varieties Y_k and Y_l . This way they assign to each (m, n) -tangle T a functor $\Psi(T)$ from the bounded derived category of equivariant coherent sheafs on Y_m to that of Y_n . They prove that for a link K the cohomology of the chain complex $\Psi(K)(\mathbb{C})$ equals the Khovanov homology of T with diagonal grading. The functors $\Psi(T)$ and our $\Phi_T^\#$ are expected to be related by mirror symmetry. Joel Kamnitzer [8] has recently proposed a method for categorifying all link polynomials from quantum groups. In this picture, for a complex reductive group G , the symplectic fibration used by Seidel and Smith (which is in fact the adjoint quotient map) is replaced by a fibration whose total space is the Beilinson-Drinfeld Grassmannian. This

Grassmannian is, roughly speaking, the moduli space of G^\vee -bundles on \mathbb{P}^1 which are trivial on the complement of a finite set of points. Here G^\vee is the Langlands dual of G . When two such points approach each other, one has a similar situation to that of Seidel-Smith where two eigenvalues come together. Kamnitzer proves a local neighborhood theorem analogous to that of Seidel and Smith.

In a forthcoming paper we show that $\mathcal{KH}_{\text{symp}}(T)$ for an (m, n) -tangle is a bimodule over (H^m, H^n) and that it is equivalent to Khovanov homology for T flat (ie. crossingless). We believe that extending the symplectic link invariant to tangles makes the comparison between the combinatorial (Khovanov) and the symplectic (Seidel-Smith) invariants easier and so serves as a small step in understanding the capability of symplectic geometry in the categorification paradigm.

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2. THE WORK OF SEIDEL AND SMITH

In this section we review the construction of Seidel and Smith which we will make use of in the rest of this paper. Most proofs are omitted. The reader is referred to [17] for details. Denote by Conf_m the space of all unordered m -tuples of distinct complex numbers (z_1, \dots, z_m) . Denote by Conf_m^0 the subset of Conf_m consisting of m -tuples which add up to zero, i.e. $z_1 + \dots + z_m = 0$.

2.1. Transverse slices. The basic reference for this section is [18]. Let G be a complex semisimple Lie group and consider the adjoint action of G on its Lie algebra \mathfrak{g} . The adjoint quotient map $\chi : \mathfrak{g} \rightarrow \mathfrak{g}/G$ sends each element of \mathfrak{g} to its orbit in \mathfrak{g}/G . A theorem of Chevalley (See [6], Chapter 23) asserts that \mathfrak{g}/G can be identified with \mathfrak{h}/W where \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} and W the associated Weyl group. Therefore χ can be regarded as assigning to each $y \in \mathfrak{g}$ the eigenvalues (or equivalently coefficients of the characteristic polynomial) of the semisimple part of y .

Definition 2.1.1. A *transverse slice* for the adjoint action at $x \in \mathfrak{g}$ is a local complex submanifold \mathcal{S} of \mathfrak{g} containing x which is transverse to the orbit of x and is invariant under the action of the isotropy subgroup G_x .

It is obvious that such an \mathcal{S} intersects the orbit of any y sufficiently close to x transversely. If K is a local submanifold of G containing the identity such that $T_e K$ is complementary to $\{y \in \mathfrak{g} : [x, y] = 0\}$ then it can be easily seen that any other transverse slice at x lies (locally) in the image of the map

$$(1) \quad \text{Ad} : K \times S \rightarrow \mathfrak{g}.$$

The Jacobson-Morozov lemma [7] tells us that if $x \in \mathfrak{g}$ is nilpotent then there are elements $y, h \in \mathfrak{g}$ such that

$$[x, y] = h \quad [x, h] = 2x \quad [y, h] = -2y.$$

Consider the vector field K on \mathfrak{g} given by $K(z) = 2z - [h, z]$. It defines a \mathbb{C}^* action on \mathfrak{g} given by $\lambda_r(z) = r^2 e^{-\log(r)} z e^{\log(r)h}$ for $r \in \mathbb{C}^*$. The vector field K vanishes at x so x is a fixed point of λ_r . A slice at x is called *homogeneous* if it is invariant under λ_r .

Now we specialize to $\mathfrak{g} = \mathfrak{sl}_{2m} = \mathfrak{sl}_{2m}(\mathbb{C})$. In this case $W = S_n$. Take x to be a nilpotent Jordan block of size $2m$. Let \mathcal{S}_m be the set of matrices in \mathfrak{sl}_{2m} of the form

$$(2) \quad \begin{pmatrix} y_1 & 1 & & & \\ y_2 & & 1 & & \\ \vdots & & & \ddots & \\ y_{n-1} & & & & 1 \\ y_n & & & & 0 \end{pmatrix}$$

where 1 is the 2×2 identity matrix, $y_1 \in \mathfrak{sl}_2$ and $y_i \in \mathfrak{gl}_2$ for $i > 1$. It is easy to see that \mathcal{S}_m is a homogeneous slice to the orbit of x . χ restricted to Conf_{2m}^0 is a differentiable fiber bundle [18]. We denote the fiber of χ over t by $\mathcal{Y}_{m,t}$, i.e. $\mathcal{Y}_{m,t} = \chi^{-1}(t)$. If $t = (\mu_1, \dots, \mu_{2m}) \notin \text{Conf}^0$, by $\mathcal{Y}_{m,t}$ we mean $\mathcal{Y}_{m,t'}$ where $t' = (\mu_1 - \sum \mu_i/2m, \dots, \mu_{2m} - \sum \mu_i/2m)$. Let E_y^μ denote the μ -eigenspace of y .

Lemma 2.1.2. *For any $y \in \mathcal{S}_m$ and $\mu \in \mathbb{C}$ the projection $\mathbb{C}^{2m} \rightarrow \mathbb{C}^2$ onto the first two coordinates gives an injective map $E_y^\mu \rightarrow \mathbb{C}^2$. Any eigenspace of any element $y \in \mathcal{S}_m$ has dimension at most two. Moreover the set of elements of \mathcal{S}_m with 2 dimensional kernel can be canonically identified with \mathcal{S}_{m-1} and this identification is compatible with χ .*

Proof. If not then the intersection of E_y^μ with $\{0\}^2 \times \mathbb{C}^{2m-2}$ is nonzero. Applying the \mathbb{C}^* action we see that the same holds for $E_{\lambda_r(y)}^{r^2\mu}$. As r goes to zero, $\lambda_r(y) \rightarrow x$ so we get $\dim \ker x > 2$ which is contradiction. From this

injectivity we see that each element of $\ker y$ is determined by its first two coordinates so if $\dim \ker y = 2$ then $y_m = 0$ and vice versa. The subset of such matrices is identified with \mathcal{S}_{m-1} . \square

For any subset $A \subset \mathfrak{sl}_{2m}$, let $A^{sub,\lambda}$ (resp. $A^{sub3,\lambda}$) be the subset of matrices in A having eigenvalue λ of multiplicity two (resp. three) and two Jordan block of size one (resp. two Jordan blocks of sizes 1,2) corresponding to the eigenvalue λ and no other coincidences between the eigenvalues. Here are two results describing neighborhoods of $\mathcal{S}_m^{sub,\lambda}$ and $\mathcal{S}_m^{sub3,\lambda}$ in \mathcal{S}_m .

Lemma 2.1.3. *Let $D \subset \text{Conf}_{2m}^0$ be a disc consisting of the $2m$ -tuples $(\mu - \epsilon, \mu - \epsilon, \mu_3, \dots, \mu_{2m})$ with ϵ small. Then there is a neighborhood U_μ of $\mathcal{S}_m^{sub,\mu}$ in $\mathcal{S}_m \cap \chi^{-1}(D)$ and an isomorphism ϕ of U_μ with a neighborhood of $\mathcal{S}_m^{sub,\mu}$ in $\mathcal{S}_m^{sub,\mu} \times \mathbb{C}^3$ such that $f \circ \phi = \chi$ on $\mathcal{S}_m \cap \chi^{-1}(D)$ where $f(x, a, b, c) = a^2 + b^2 + c^2$. Also if $N_y \mathcal{S}_m^{sub,\mu}$ denotes the normal bundle to $\mathcal{S}_m^{sub,\mu}$ at y the we have*

$$(3) \quad \phi(N_y \mathcal{S}_m^{sub,\mu}) = \mathfrak{sl}(E_y^\mu) \oplus \zeta_y$$

where ζ_y is the trace free part of $\{\mathbb{C} \subset \mathfrak{gl}(E_y^\mu)\} \oplus \mathfrak{gl}(E_y^{\mu_3}) \oplus \dots \oplus \mathfrak{gl}(E_y^{\mu_{2m}})$.

Proof. For $y \in \mathcal{S}_m^{sub,\mu}$, let \mathcal{S}_y be a subspace of $T_y \mathcal{S}_m$ complementary to $T_y \mathcal{S}_m^{sub,\lambda}$ which depends holomorphically on y . These subspaces together form a tubular neighborhood of $\mathcal{S}_m^{sub,\mu}$ in \mathcal{S}_m . Since \mathcal{S}_m and $\mathfrak{sl}_{2m}^{sub,\mu}$ intersect transversely, \mathcal{S}_y is also a transverse slice at y for the adjoint action on \mathfrak{sl}_{2m} . We can produce another family of transverse slices \mathcal{S}'_y by setting $\mathcal{S}'_y = \mathfrak{sl}(E_y^\mu) \oplus \zeta_y$ which equals the trace free part of $\mathfrak{gl}(E_y^\mu) \oplus \mathfrak{gl}(E_y^{\mu_3}) \oplus \dots \oplus \mathfrak{gl}(E_y^{\mu_{2m}})$. The reason is that $[y, \mathfrak{sl}_{2m}] = 0 \oplus \mathfrak{sl}_{2m-2}^0$ where the first component consists of zero in \mathfrak{sl}_2 and the second one consists of matrices with zeros on the diagonal and the right hand side is transverse to \mathcal{S}'_y .

Now \mathcal{S}_y is isomorphic (as local complex manifolds) to \mathcal{S}'_y for each y with an isomorphism that moves points only inside their adjoint orbits (and hence is compatible with χ). We can choose these isomorphisms to depend holomorphically on y . Each $\mathfrak{gl}(E_y^{\mu_i}) \subset \mathfrak{sl}_{2m}$ for $i > 2$ can be canonically (without choice of a basis) identified with \mathbb{C} . Lemma 2.1.2 tells us that E_y^μ can be canonically identified with \mathbb{C}^2 so $\mathfrak{sl}(E_y^\mu)$ is identified with \mathfrak{sl}_2 . It follows that $\mathcal{S}'_y \cong \mathfrak{sl}_2 \oplus \mathbb{C}^{2m-2}$. The desired ϕ is the composition of the two isomorphisms in this paragraph. \square

Remark 2.1.4. If y has two linearly independent μ_1 eigenvectors as well as two linearly independent μ_2 eigenvectors and with no other coincidences

between the eigenvalues, we can repeat the above argument to obtain

$$(4) \quad \phi(N_y(\mathcal{S}_m^{sub,\mu_1} \cap \mathcal{S}_m^{sub,\mu_2})) = \mathfrak{sl}(E_{\mu_1}) \oplus \mathfrak{sl}(E_{\mu_2}) \oplus \zeta.$$

So ϕ gives an isomorphism between a neighborhood of $\mathcal{S}_m^{sub,\mu_1} \cap \mathcal{S}_m^{sub,\mu_2}$ in \mathcal{S}_m and $(\mathcal{S}_m^{sub,\mu_1} \cap \mathcal{S}_m^{sub,\mu_2}) \times \mathbb{C}^3 \times \mathbb{C}^3$.

Consider the line bundle \mathcal{F} on $\mathcal{S}_m^{sub3,\mu}$ whose fiber at $y \in \mathcal{S}_m^{sub3,\mu}$ is $(y - m)E_{y_s}^\mu$ where y_s is the semisimple part of y . To \mathcal{F} one associates a \mathbb{C}^4 bundle $\mathcal{L} = (\mathcal{F} \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4$ where $z \in \mathbb{C}^*$ acts on \mathbb{C}^4 by

$$(5) \quad (a, b, c, d) \rightarrow (a, z^{-2}b, z^2c, d).$$

\mathcal{L} decomposes as

$$(6) \quad \mathcal{L} \cong \mathbb{C} \oplus \mathcal{F}^{-2} \oplus \mathcal{F}^2 \oplus \mathbb{C}.$$

Fibers of \mathcal{L} should be regarded as transverse slices in \mathfrak{sl}_3 . Upon choosing suitable coordinates on such a transverse slice (at the zero matrix) the function χ equals the function $p : \mathfrak{sl}_3 \rightarrow \mathbb{C}^2$ given by

$$(7) \quad p(a, b, c, d) = (d, a^3 - ad + bc).$$

p is also well-defined as a function $\mathcal{L} \rightarrow \mathbb{C}^2$ because b and c are coordinates on line bundles which are inverses of each other. Denote by $\tau(d, z)$ the set of solutions of $\lambda^3 - d\lambda + z = 0$.

Lemma 2.1.5. *Let $P \subset \text{Conf}_{2m}^0$ be the set of $2m + 2$ -tuples*

$$(8) \quad (\mu_1, \dots, \mu_{i-1}, \tau(d, z), \mu_{i+3}, \dots, \mu_{2m+2}).$$

where d and z vary in a small disc in \mathbb{C} containing the origin. There is a neighborhood V of \mathcal{S}_m^{sub3} in $\mathcal{S}_m \cap \chi^{-1}(P)$ and an isomorphism ϕ' from V to a neighborhood of zero section in \mathcal{L} such that $p(\phi'(x)) = (d, z)$ if

$$\chi(x) = (\mu_1, \dots, \mu_{i-1}, \tau(d, z), \mu_{i+3}, \dots, \mu_{2m+2}).$$

2.2. Relative vanishing cycles. Let X be a complex manifold and K a compact submanifold. Let g be a Kahler metric on $Y = X \times \mathbb{C}^3$ (not necessarily the product metric) and denote its imaginary part by Ω . Consider the map $f : X \times \mathbb{C}^3 \rightarrow \mathbb{C}$ given by $f(x, a, b, c) = a^2 + b^2 + c^2$ and denote by ϕ_t the gradient flow of $-\text{Re } f$. Let W be the set of points $y \in Y$ for which the trajectory $\phi_t(y)$ exists for all positive t .

Lemma 2.2.1. *W is a manifold and l is smooth. The function $l : W \rightarrow X$ given by $l(y) = \lim_{t \rightarrow \infty} \phi_t(y)$ is well-defined and smooth. We have $\Omega|_W = l^*\Omega|_X$. f restricted to W is real and nonnegative.*

Proof. The first two assertions follow from stable manifold theorem (Theorem 1 in [5]) and the rest from the fact that gradient vector field of $-\operatorname{Re} f$ is the Hamiltonian vector field of $\operatorname{Im} f$. \square

Set $V_t(K) = \pi^{-1}(t) \cap l^{-1}(K) = l|_{\pi^{-1}(t)}^{-1}(K)$ which is a manifold for t small. It follows from Morse-Bott lemma that $V_t(K)$ is a 2-sphere bundle on K for t small. To generalize the invariant to tangles we will need a slightly more general version of the above construction in which K is noncompact and the metric equals the product metric outside a compact subset. (See section 4.2.) The resulting vanishing cycle equals (symplectically) the product bundle outside a compact subset.

2.3. Fibered A_2 singularities. Assume we have the same situation as in the Lemma 2.1.5, i.e. let \mathcal{F} be a holomorphic line bundle over a complex manifold X and define Y to be $(\mathcal{F} \setminus 0) \times_{\mathbb{C}^*} \mathbb{C}^4$ where the \mathbb{C}^* action is as in the formula (5). Let Ω be an arbitrary Kahler form on Y and by regarding X as the zero section of Y , Ω restricts to a Kahler form on X . Let (a, b, c, d) be the coordinates on fibers of $Y \rightarrow X$ and (d, z) coordinates on \mathbb{C}^2 . Let the map $p : Y \rightarrow \mathbb{C}^2$ be as in Lemma 2.1.5. Let $Y_d = p^{-1}(\mathbb{C} \times \{d\})$ and $p_d : Y_d \rightarrow \mathbb{C}$ be the restriction of p . Set $Y_{d,z} = p^{-1}(d, z)$. For $d \neq 0$ critical values of p_d are $\zeta_d^\pm = \pm 2\sqrt{d^3/27}$.

Let K be Lagrangian submanifold of X . Using relative vanishing cycle construction for the function p_d we can obtain a Lagrangian submanifold L_d of Y which is a sphere bundles over K . (This construction works when Y is a nontrivial bundle over X as well.) There is another way of describing this Lagrangian as follows. Let $Y \cong \mathbb{C}^4$ be the fiber of $Y \rightarrow X$ over some point of X and let $p : Y \rightarrow \mathbb{C}^2$ be as before. The restriction of the \mathbb{C}^* action to S^1 is a Hamiltonian action with the moment map $\mu(a, b, c, d) = |c|^2 - |b|^2$. Define

$$(9) \quad C_{d,z,a} = \{(b, c) : \mu(b, c) = 0, a^3 - da - z = -bc\} \subset Y_{d,z}$$

which is a point if $a^3 - da - z = 0$ and a circle otherwise. The three solutions of this equation correspond to the critical values of the projection $q_{d,z} : Y_{d,z} \rightarrow \mathbb{C}$ to the a plane. In the situation of Lemma 2.1.5 they correspond to the three eigenvalues of Y . Let $\alpha(r)$ be any embedded curve in \mathbb{C} which intersects these critical values (only) if $r = 0, 1$. Define

$$(10) \quad \Lambda_\alpha = \bigcup_{r=0}^1 C_{d,z,\alpha(r)}$$

which is a Lagrangian submanifold of $Y_{d,z}$ (with Kahler form induced from \mathbb{C}^4). Let $\mathbf{c}, \mathbf{c}', \mathbf{c}''$ be as in the Figure 1 where dots represent the critical

values of $q_{d,z}$. We can associate to K a Lagrangian submanifold $\Lambda_{d,\alpha}$ of Y by defining $\Lambda_{d,\alpha} = (Y|K) \times_{S^1} \Lambda_{\mathbf{c}}$. Seidel and Smith prove that these two procedures give the same result:

Lemma 2.3.1. *If the Kahler form on Y is obtained from a Kahler form on X , a Hermitian metric on \mathcal{F} and the standard form on \mathbb{C}^4 then $L_d = \Lambda_{d,\mathbf{c}}$.*

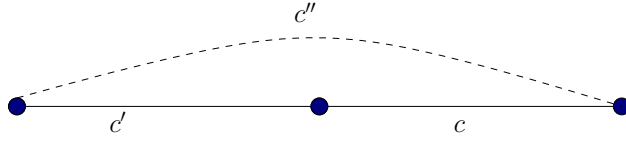


FIGURE 1.

2.4. Symplectic structure. The symplectic structure that Seidel and Smith use on \mathcal{S}_m is not the standard structure on $\mathcal{S}_m \cong \mathbb{C}^{4m-1}$. The reason is to obtain well-defined parallel transport maps for the fibration χ whose fibers are noncompact. We need a plurisubharmonic function whose fiberwise critical point set project properly under χ . Fix $\alpha > m$. Let $\xi_i(z) = |z|^{2\alpha/i}$ for $z \in \mathbb{C}$. These functions are C^1 however by adding compactly supported functions η_i we can obtain C^∞ functions $\psi_i = \xi_i + \eta_i$ on \mathbb{C} . We choose η_i such that $-dd^c\psi_i > 0$. Let ψ be the function on \mathcal{S}_m whose value at $y \in \mathcal{S}_m$ is

$$\sum_i \sum_{\mu, \nu \in \{0,1\}} \psi_i((y_{1i})_{\mu, \nu}).$$

We can choose η_i so that ψ is an exhausting plurisubharmonic function on \mathcal{S}_m which gives us the symplectic form $\Omega = -dd^c\psi$. Outside a set, which is the product of the complement of a compact set in each coordinate plane, we have

$$\Omega = 4 \sum_i (\alpha/i)(\alpha/i - 1) \sum_{\mu, \nu \in \{0,1\}} |(y_{1i})_{\mu, \nu}|^{2\alpha/i} d(y_{1i})_{\mu, \nu} \wedge \overline{d(y_{1i})_{\mu, \nu}}.$$

By restriction we obtain Stein structures on each $\mathcal{Y}_{m,t}$. The addition of the functions η_i prevents ϕ from being homogeneous with respect to the λ_r action but as $r \rightarrow \infty$ the functions $\eta_i(r^i z)$ are supported on smaller and smaller neighborhood of origin so $\eta_i(r^i z)/r^{2\alpha}$ go to zero and so we get the asymptotic homogeneity of ψ , i.e.:

$$(11) \quad \lim_{r \rightarrow \infty} \frac{\psi \circ \lambda_r}{r^{2\alpha}} = \xi.$$

Since the fibers $\mathcal{Y}_{m,t}$ are noncompact, existence of parallel transport maps for the fiber bundle $\chi|_{\text{Conf}_{2m}^0}$ is not guaranteed. Let β be a curve in Conf_{2m} and H_β be the horizontal lift of $\dot{\beta}$ and $Z_{\beta(s)}$ be the projection of $\nabla\psi(\beta(s))$ to $\mathcal{Y}_{m,\gamma(s)}$. Seidel and Smith obtain a *rescaled parallel transport* map $h_\beta^{\text{res}} : \mathcal{Y}_{m,\beta(0)} \rightarrow \mathcal{Y}_{m,\beta(1)}$ which is given by integrating the vector field

$$(12) \quad H_\beta - \sigma Z_\beta$$

and then composing by the time σ map of $Z_{\beta(1)}$ where σ is a positive constant (depending on β). h_β^{res} is a symplectomorphism defined on arbitrarily large compact subsets of $\mathcal{Y}_{m,\beta(0)}$. For this procedure to work, one needs the fiberwise critical point set of ψ to project properly under χ and the homogeneity property (11) ensures this. See [17] Section 5A.

If $\mu \in \mathbb{C}^{2m}/\mathbf{S}_{2m}$ has only one element of multiplicity two or higher, which we denote by μ_1 , denote by $\mathcal{D}_{m,\mu}$ the set of singular elements of $(\chi^{-1}(\mu) \cap \mathcal{S}_m)$ i.e.

$$(13) \quad \mathcal{D}_{m,\mu} = (\chi^{-1}(\mu) \cap \mathcal{S}_m)^{\text{sub},\mu_1}.$$

Let \mathcal{D}_m be the union of all these $\mathcal{D}_{m,t}$ regarded as a subset of \mathcal{S}_m . It inherits a Kahler metric from \mathcal{S}_m . We have the map $\chi : \mathcal{D}_m \rightarrow \mathbb{C} \times \mathbb{C}^{2m-2}/\mathbf{S}_{2m-2}$. By forgetting the first eigenvalue, the image of χ can be identified with Conf_{2m-2} . Lemma 2.1.2 tells us that if $\mu_1 = 0$ then $\mathcal{D}_{m,\mu}$ can be identified with $\mathcal{Y}_{m-1,\mu \setminus \{\mu_1\}}$. It can be shown ([17] Section 5A) that χ is a differentiable fiber bundle and we have rescaled parallel transport maps

$$(14) \quad h_\beta^{\text{res}} : \mathcal{D}_{m,\beta(0)} \rightarrow \mathcal{D}_{m,\beta(1)}$$

for any curve β in Conf_{2m-2} . These parallel transport maps are compatible with those for $\mathcal{Y}_{m-1,\mu \setminus \{0\}}$ under the identification above provided that β lies in Conf_{2m-2}^0 . This is because of the special (product) form of the symplectic structure.

2.5. Lagrangian submanifolds from crossingless matchings. Let $\mu \in \text{Conf}_{2m}$. A *crossingless matching* D with endpoints in μ is a set of m disjoint embedded curves $\delta_1, \dots, \delta_m$ in \mathbb{C} which have (only) elements of μ as endpoints. See Figure 2. To D we associate a Lagrangian submanifold L_D of $\mathcal{Y}_{m,\mu}$ as follows. Let $\{\mu_{2k-1}, \mu_{2k}\} \subset \mu$ be the endpoints of δ_k for each k . Let γ be a curve in Conf_m^0 such that $\gamma(t) = \{\gamma_1(t), \gamma_2(t), \mu_3, \mu_4, \dots, \mu_{2m}\}$, $\gamma_i(0) = \mu_i, i = 1, 2$ and as $s \rightarrow 1$, $\gamma_1(t), \gamma_2(t)$ approach each other on δ_1 and collide. For example if $\delta_1(t)$ is a parametrization of δ_1 s.t $\delta_1(0) =$

$\mu_1, \delta_1(1) = \mu_2$ then we can take $\gamma(t) = \{\delta(t/2), \delta(1 - t/2), \mu_3, \dots, \mu_{2m}\}$. Set $\bar{\mu} = \mu \setminus \{\mu_1, \mu_2\}$, $\mu' = \gamma(1)$.

If $m = 1$ then relative vanishing cycle construction for $\chi : \mathcal{S}_1 \rightarrow \mathbb{C}$ with the critical point over $\gamma(1) = 0$ gives us a Lagrangian sphere L in $\mathcal{Y}_{1, \gamma(1-s)}$ for small s . Using reverse parallel transport along γ we can move L to $\mathcal{Y}_{1, \mu}$ to get our desired Lagrangian submanifold. Now for arbitrary m assume by induction that we have obtained a Lagrangian $L_{\bar{D}} \subset \mathcal{Y}_{m-1, \bar{\mu}}$ for \bar{D} which is obtained from D by deleting δ_1 . Now $\mathcal{Y}_{m-1, \bar{\mu}}$ can be identified with $D_{m, \tau}$ where $\tau = (0, 0, \mu_3 - (\mu_1 + \mu_2)/(2m - 2), \dots, \mu_{2m} - (\mu_1 + \mu_2)/(2m - 2))$. Use parallel transport to move $L_{\bar{D}}$ to $D_{m, \gamma(1)}$. The later one is the set of singular points of $\mathcal{Y}_{m, \gamma(1)}$ so Lemma 2.1.3 tells us that we can use relative vanishing cycle construction for $L_{\bar{D}}$ to obtain a Lagrangian in $\mathcal{Y}_{m, \gamma(1-s)}$ for small s . Parallel transporting it along γ back to $\mathcal{Y}_{m, \mu}$ we obtain our desired Lagrangian which is topologically a trivial sphere bundle on $L_{\bar{D}}$. We see that L_D is diffeomorphic to a product of spheres. Different choices of the curve γ result in Hamiltonian isotopic Lagrangians. The same holds if we isotope the curves in D inside $\mathbb{C} \setminus \mu$.



FIGURE 2. Two crossingless matchings

2.6. Grading. The topic of this subsection is the absolute grading of Floer cohomology groups. We discuss the special case which is of concern in this paper, i.e. the case of \mathbb{Z} grading. The reader is referred to [16] for more detail. Let (M^{2n}, ω) be a symplectic manifold. We know that for a Lie group G and its maximal compact subgroup K , the isomorphism classes of G bundles on a topological space X are in one-to-one correspondence with those of K bundles over X . The symplectic group $Sp(2n)$ has $U(n)$ as maximal compact subgroup therefore upon choosing a compatible almost complex structure the structure group of TM can be reduced to $U(n)$. Denote by P the principal $U(n)$ bundle we get from TM in this way.

Definition 2.6.1. An *infinite Maslov cover* of M is a $SU(n)$ bundle \tilde{P} over M such that $\tilde{P} \times_{SU(n)} \mathbb{C}^n = TM$.

Since the universal cover $\widetilde{U(n)}$ of $U(n)$ has $SU(n)$ as maximal compact subgroup, we get an equivalent definition by replacing $SU(n)$ with $U(n)$ in the above definition.

Lemma 2.6.2. *If the structure group of TM can be further reduced to $SU(n)$ then M has an infinite Maslov covering.*

Proof. Let $g_{\alpha,\beta}$ and $g'_{\alpha,\beta}$ be transition functions of TM as a $U(n)$ resp. $SU(n)$ bundle over the same covering of M . Then the transition functions $\{(g_{\alpha,\beta}, t) \mid g_{\alpha,\beta} = e^{it} g'_{\alpha,\beta}\}$ define a $\widetilde{U(n)}$ bundle \tilde{P} over M for which we have $\tilde{P} \times_{\widetilde{U(n)}} \mathbb{C}^n = TM$. □

The isomorphism classes of such covers are in one to one correspondence with $H_1(M)$. Let $\mathcal{L} \rightarrow M$ be the bundle whose fiber at $x \in M$ is the Lagrangian Grassmanian of $T_x M$. A Maslov cover \tilde{P} , induces a covering $\tilde{\mathcal{L}} \rightarrow \mathcal{L}$ given by $\tilde{P} \times_{SU(n)} \widetilde{Lag}(n)$. Each Lagrangian submanifold L of M determines a section s_L of \mathcal{L} . A *grading* of L is a cover \tilde{s}_L of s_L . If L_0, L_1 are two Lagrangian submanifolds of M that intersect transversely we can assign an absolute grading to each intersection point x as follows. Let $\tilde{\Lambda}_i = \tilde{s}_{L_i}$ for $i = 0, 1$. Let $\tilde{\Lambda}(t)$ be a path joining $\tilde{\Lambda}_0$ to $\tilde{\Lambda}_1$ and let $\Lambda(t)$ be its projection. Define

$$(15) \quad \deg(x) = \mu^{path}(\Lambda, T_x L_0) + \frac{n}{2}$$

where μ^{path} is the Maslov index for paths. It does not depend on the choice of the liftings \tilde{s}_{L_i} because of the naturality of μ^{path} . In general if the canonical line bundle is not trivial, one can obtain only a \mathbb{Z}/N grading for some $N \in \mathbb{N}$.

There is an equivalent way of describing this grading. If the canonical bundle of M is trivial, it has a global section (or trivialization) η . The global section η gives us a map $\alpha_M : M \rightarrow S^1$ given by

$$(16) \quad \alpha_M(x) = \frac{\eta(u_1, \dots, u_n)^2}{|\eta(u_1, \dots, u_n)|^2}$$

for any orthonormal basis u_1, \dots, u_n of $T_x M$. For each Lagrangian submanifold L we can define a *phase function* $\alpha_L : L \rightarrow S^1$ by

$$(17) \quad \alpha_L(x) = \frac{\eta(v_1, \dots, v_n)^2}{|\eta(v_1, \dots, v_n)|^2}$$

for any orthonormal basis v_1, \dots, v_n of $T_x L$.

Alternative Definition 2.6.3. A *grading* of L is a choice of a real valued function \tilde{L} such that $\alpha_L = \exp(2\pi i \tilde{L})$.

For a pair of transversely intersecting graded Lagrangians L_0, L_1 and $x \in L_0 \cap L_1$ one can set

$$(18) \quad \deg(x) = \frac{n}{2} + \tilde{L}_0(x) - \tilde{L}_1(x).$$

We denote by $L\{m\}$, L with its grading shifted by m , ie. $\tilde{L}\{m\} = \tilde{L} - m$. A grading for a diffeomorphism ϕ from M to itself is a choice of a function $\tilde{\phi} : M \rightarrow \mathbb{R}$ such that $\exp(2\pi i \tilde{\phi}(x)) = \alpha_M(x)/\alpha_M(\phi^{-1}(x))$. $\phi(L)$ has a preferred grading given by

$$(19) \quad \widetilde{\phi(L)}(x) = \tilde{L}(\phi^{-1}(x)) + \tilde{\phi}(\phi^{-1}(x)).$$

A choice of grading for ϕ induces a grading on the graph Γ of ϕ :

$$(20) \quad \tilde{\Gamma}(x, \phi(x)) = \tilde{\phi}(x).$$

Since each manifold $\mathcal{Y}_{m,\nu}$ is a submanifold of the affine space \mathcal{S}_m and has trivial normal bundle, its Chern classes are zero. This together with the fact that $H_1(\mathcal{Y}_m) = 0$ implies that the canonical bundle of $\mathcal{Y}_{m,\nu}$ is trivial and so has a unique infinite Maslov cover. We start by choosing global sections $\eta_{\mathcal{S}_m}$ and $\eta_{\mathfrak{h}/W}$. Then we choose trivializations for regular fibers of $\chi_{\mathcal{S}_m}$ characterized by $\eta_{\mathcal{Y}_{m,t}} \wedge \chi^* \eta_{\mathfrak{h}/W} = \eta_{\mathcal{S}_m}$. If we choose a grading for $L \subset \mathcal{Y}_{m,t_0}$ and β is a curve in Conf_{2m} starting at t_0 , one can continue the grading on L uniquely to $h_{\beta|_{[0,s]}}(L)$ for any s . Therefore the grading of L uniquely determines that of $h_\beta(L)$.

2.7. The invariant. Now we can define the Seidel-Smith invariant. The definition uses Floer cohomology which we review (in a bit more general setting) in section 3.3. Let \mathcal{D}_+ be the crossingless matching at the left hand side of picture 2. If a link K is obtained as closure of a braid $\beta \in Br_m$, let $\beta' \in \text{Conf}_{2m}$ be the braid obtained from β by adjoining the identity braid id_m .

Definition 2.7.1.

$$\mathcal{H}_{\text{sympl}}^*(K) = HF^{*+m+w}(L_{\mathcal{D}_+}, h_{\beta'}^{\text{res}}(L_{\mathcal{D}_+}))$$

Here w is the writhe of the braid presentation, i.e. the number of positive crossings minus the number of the negative crossings in the presentation. Since the manifold is convex at infinity and the Lagrangians are exact, the above Floer cohomology is well-defined. Independence from choice of β is proved in [17], section 5C. Well-definedness of the invariant developed in the present paper gives an alternative proof. See Theorem 4.2.8.

3. STEIN SYMPLECTIC VALUED FIELD THEORIES

In this section we generalize symplectic valued field theories to allow a class of Lagrangian submanifolds of Stein manifolds.

3.1. Some remarks on Lagrangian submanifolds of Stein manifolds. Let (M, ψ) be a Stein manifold. This means that ψ is proper, bounded below and $-dd^c\psi$ is a symplectic form on M . For a subset $N \subset M$, we denote by $N_{\leq c}$ (N_c) the intersection of N with sublevel (level) sets of ψ . Also set $\theta = -d^c\psi$.

Definition 3.1.1. A Lagrangian submanifold $L \subset M$ is called *c-allowable* if it is exact and $\psi|_L$ does not have any critical points in $M_{\geq c}$. It is called *allowable* if it is *c-allowable* for some c . In addition any compact monotone Lagrangian submanifold of any symplectic manifold is considered allowable.

Note that this implies that L intersects the level sets of ψ transversely at infinity. Compact Lagrangian submanifolds of Stein manifolds are evidently allowable. Let $Z = \nabla\psi$ be the Liouville vector field on M . Denote by $\kappa : \mathbb{R} \times M \rightarrow M$ the flow of Z .

Definition 3.1.2. A Lagrangian L in a Stein manifold M is said to have a *conical end* if there is a constant c such that L intersects M_c transversely and $\kappa([0, \infty) \times (L \cap M_c))$ equals $L_{\geq c}$.

An exact Lagrangian with conical end is clearly allowable. Our next task is to assign a Lagrangian with conical end to an allowable Lagrangian which can replace the former when Floer cohomology is concerned. We first need some definitions.

Definition 3.1.3. A Lagrangian submanifold Λ of the symplectic manifold $M_{\leq c}$ is called *κ -compatible* if it intersects M_c transversely (possibly empty) and there is an $\epsilon > 0$ such that $\kappa([-\epsilon, 0] \times \Lambda_c) = ({}_{c-\epsilon}\Lambda_{\leq c})$.

Definition 3.1.4. A Hamiltonian isotopy induced by a time-dependent function H_t is called *conical* if there is a constant c such that $H_t \circ \kappa(r, x) = rH_t(\kappa(0, x))$ for all $r \geq 0$, $t \in [0, 1]$ and x in $M_{\geq c}$.

Let ϕ_t be a one parameter family of symplectomorphisms of M and $L \subset M$ a Lagrangian submanifold. The isotopy $\phi_t(L)$ is called *exact* if $\phi_t^*\theta|_L = \theta|_L + dK_t$ for any $t \in [0, 1]$ where K_t is a function on L depending smoothly on t . We have the following facts from [11] Section 5.

Lemma 3.1.5. *i) Any exact Lagrangian in $M_{\leq c}$ which intersects the boundary transversely can be exact-isotoped rel boundary in $M_{\leq c}$ to a κ -compatible one.*

ii) If Λ_t is a Lagrangian isotopy in $M_{\leq c}$ such that all Λ_t intersect the boundary transversely and Λ_0, Λ_1 are κ -compatible then there is another isotopy Λ'_t with the same endpoints such that $\Lambda_t \cap M_c = \Lambda'_t \cap M_c$ for any $t \in [0, 1]$ and all Λ_t are κ -compatible. If Λ_t is exact, Λ'_t can be chosen to be so.

We include the proof for completeness.

Proof. [11], Lemma 5.2 i) Let L be such a Lagrangian. Choose $r < c$ such that $K = L_{\geq -r}$ deformation retracts onto ∂L . $\theta|_K$ is closed and vanishes on the boundary so $\theta|_K = df$ for some f on K which vanishes on boundary. Extend f to a smooth function h on $M_{\leq c}$ vanishing on the boundary and set $\theta_t = \theta - t dh$ for $t \in [0, 1]$. Let Z_t be a vector field such that $\iota_{Z_t} \omega = \theta_t$. Since $d\theta_t = \omega$, Z_t is symplectic and defines an embedding $\kappa_t : M_c \times [0, \infty] \rightarrow M_{\leq c}$. Let U be a neighborhood of M_c which lies in the image of κ_t for all t and deformation retracts onto M_c . Set $X_t = \partial \kappa_t / \partial t \circ (\kappa_t)^{-1}|_U$. It is a symplectic vector field which vanishes on ∂M . $\iota_{X_t} \omega$ is closed and vanishes on boundary so equals dH_t for some H_t on U vanishing on π_c . Each X_t can be extended to a Hamiltonian vector field \tilde{X}_t on $M_{\leq c}$. Let ϕ_t be the flow of the time dependent vector field (\tilde{X}_t) . We have $\phi_t \circ \kappa_0 = \kappa_t$ on a neighborhood of M_c so $\phi_1^* \theta = (\kappa_0^{-1})^* \kappa_1^* \theta = \theta_1$ near M_c . Therefore

$$(21) \quad \phi_1^*(\theta|_L) = \theta_1|_L = 0$$

near the boundary of L and so $\phi_1(L)$ is κ -compatible. By (21), $(\phi_t(L))$ is exact.

ii) It is just a parameterized version of i).

□

Lemma 3.1.6. *Any exact Lagrangian isotopy rel boundary in $M_{\leq c}$ which consists only of κ -compatible Lagrangians can be embedded into a Hamiltonian isotopy.*

The proof is similar to the proof of the fact that a symplectomorphism of zero flux is Hamiltonian.

Definition 3.1.7. Let L be a c -allowable Lagrangian in M , we take $\Lambda = L_{\leq c}$, isotope Λ to a κ -compatible Λ' and denote by $\mathfrak{C}_c(L) \subset M$ the Lagrangian with conical end associated to Λ' . This means that $\mathfrak{C}_c(L)_{>c}$ is the image of L_c under the Liouville flow.

It follows from the above lemmas that $\mathfrak{C}_c(L)$ is well-defined up to conical Hamiltonian isotopy. Moreover if $L_{\leq c}$ and $L'_{\leq c}$ are exact isotopic rel boundary then $\mathfrak{C}_c(L)$ and $\mathfrak{C}_c(L')$ are conical Hamiltonian isotopic.

3.2. Symplectic valued topological field theories. A *Lagrangian correspondence* between two symplectic manifolds M_0 and M_1 is a Lagrangian submanifold of $M_0^- \times M_1$. If $L_{0,1}$ is a Lagrangian correspondence between M_0, M_1 and $L_{1,2}$ is a correspondence between M_1, M_2 then the *composition* $L_{0,1} \circ L_{1,2}$ is defined as

$$L_{0,1} \circ L_{1,2} := \{(m, m'') | \exists m' \in M_1 \text{ s.t. } (m, m') \in L_{0,1}, (m', m'') \in L_{1,2}\}$$

which is a subset of $M_0 \times M_2$.

Definition 3.2.1. This composition is *embedded* if $L_{0,1} \times L_{1,2} \subset M_0 \times M_1 \times M_2$ intersects the diagonal $M_0 \times \Delta_{M_1} \times M_2$ transversely and the projection $\pi_{0,2}$ embeds the intersection into $M_0 \times M_2$. In this case the composition is a Lagrangian submanifold of $M_0^- \times M_2$.

If M_i , $i = 0, 1, 2$ have trivial canonical bundle and we have chosen trivializations η_{M_i} for each i then gradings $\tilde{L}_{0,1}$ and $\tilde{L}_{1,2}$ determine a grading on $L_{0,2} = L_{0,1} \circ L_{1,2}$, with regard to the trivialization $\eta_{M_0} \wedge \eta_{M_2}$, given by

$$(22) \quad \tilde{L}_{0,2}(m, m'') = \tilde{L}_{1,2}(m, m') + \tilde{L}_{0,1}(m', m'')$$

where m' is the unique point such that $(m, m', m', m'') \in (L_{0,1} \times L_{1,2}) \cap M_0 \times \Delta_{M_1} \times M_2$ provided that the composition is embedded. If the M_i are Stein and the Lagrangian correspondences have conical ends then it is easy to see that their composition has a conical end as well. Therefore we have the following.

$$(23) \quad \mathfrak{C}_c(L_{0,1}) \circ \mathfrak{C}_c(L_{1,2}) \simeq \mathfrak{C}_c(L_{0,1} \circ L_{1,2})$$

Definition 3.2.2. A *generalized Lagrangian correspondence* between symplectic manifolds M, M' consists of a sequence $M = M_0, M_1, \dots, M_n = M'$ of symplectic manifolds and a sequence $\underline{L} = (L_{0,1}, \dots, L_{i,i+1}, \dots, L_{n-1,n})$ such that $L_{i,i+1}$ is a Lagrangian correspondence between M_i and M_{i+1} . Generalized Lagrangian correspondences can be composed by concatenation. We call \underline{L} *compact* if each $L_{i,i+1}$ is compact. It is called *allowable* if all the $L_{i,i+1}$ are so.

Definition 3.2.3. We denote the composition (concatenation) of \underline{L} and \underline{L}' by $\underline{L} \# \underline{L}'$. Suppose all the manifolds M_i involved have trivial canonical bundle and we have chosen trivializations η_{M_i} for each M_i . A *grading* on \underline{L} is a lift $\tilde{L}_{i,i+1}$ of $\alpha_{L_{i,i+1}}$ for each i where the phase functions $\alpha_{L_{i,i+1}}$ are with respect to $\eta_{M_i}^- \wedge \eta_{M_{i+1}}$.

Assume we have chosen a trivialization η_M for the canonical bundle of M . The canonical bundle of M^- is the dual of $\bigwedge^{\dim M} TM$. Thus we can take $\sqrt{-1}\eta^{-1}$ as a trivialization of the determinant bundle of M^- . So the

phase function of a Lagrangian $L \subset M$ is the negative of the inverse of the phase function of the same Lagrangian as a subset of M^- . (We denote the later one by L_- .) Therefore a grading \tilde{L} of L as a Lagrangian submanifold of M induces, in a natural way, a grading of L as a submanifold of M^- . This new grading equals $k/2 - \tilde{L}$ for some integer k . We choose $k = n = \dim L$ therefore we have

$$(24) \quad \tilde{L}_- = \frac{n}{2} - \tilde{L}.$$

According to [19] Section 2.2, the *symplectic category* is the category whose objects are compact monotone symplectic manifolds (including exact ones) and whose morphisms are equivalence classes of compact generalized Lagrangian correspondences. The equivalence relation on morphisms is generated by the following two relations. Firstly,

$$(L_0, L_{0,1}, \dots, L_{i,i+1}, \dots, L_{n-1,n}, L_n)$$

is equivalent to

$$(L'_0, L'_{0,1}, \dots, L'_{i,i+1}, \dots, L'_{n-1,n}, L'_n)$$

if each $L_{i,i+1}$ is Hamiltonian isotopic to $L'_{i,i+1}$ in $M_i^- \times M_{i+1}$. Secondly

$$(L_0, L_{0,1}, \dots, L_{i-1,i}, L_{i,i+1}, \dots, L_n)$$

is equivalent to

$$(L_0, L_{0,1}, \dots, L_{i-1,i} \circ L_{i,i+1}, \dots, L_n)$$

whenever the composition $L_{i-1,i} \circ L_{i,i+1}$ is embedded. The idea of symplectic category goes back to Weinstein [20] where he considered only Lagrangian correspondences as morphisms. Since the composition of two Lagrangian correspondences might not be embedded, he did not obtain a genuine category.

Definition 3.2.4. A $d + 1$ dimensional *symplectic valued topological field theory* is a functor from the category (d -manifolds, cobordisms) to the symplectic category. Here “cobordism” means cobordism modulo isotopy.

In the present paper we need to include a class of noncompact symplectic manifolds and Lagrangians so we enlarge the symplectic category.

Definition 3.2.5. The *noncompact symplectic category* has objects of the symplectic category plus Stein manifolds as objects. Morphisms are given by equivalence classes of generalized Lagrangian correspondences $(L_{0,1}, \dots, L_{n-1,n})$ such that each $L_{i,i+1}$ is allowable. The equivalence relation on correspondences is the one in the symplectic category with the

difference that we restrict the first kind of equivalence of morphisms to include only exact isotopies.

3.3. Floer cohomology. Let \underline{L} be a generalized Lagrangian correspondence as in the last section. By adding a trivial Lagrangian correspondence (i.e. the diagonal) if necessary we can assume that $n = 2k + 1$ is odd. Define

$$(25) \quad \mathcal{L}_0 = L_0 \times L_{1,2} \times \cdots \times L_{2k-1,2k}$$

and

$$(26) \quad \mathcal{L}_1 = L_{0,1} \times L_{2,3} \times \cdots \times L_{2k+1}$$

which are Lagrangian submanifolds of $M = M_0^- \times M_1 \times \cdots \times M_n^-$. One can, under good circumstances, associate to \underline{L} the Floer cohomology group

$$(27) \quad HF(\underline{L}) := HF(\mathfrak{C}_c(\mathcal{L}_0), \mathfrak{C}_c(\mathcal{L}_1))$$

for some c large enough. Here we briefly review the construction emphasizing the aspects that are of concern in this paper. See [19] for more details in the case of compact Lagrangians. First we assume that there is a conical Hamiltonian isotopy of M which makes \mathcal{L}_0 and \mathcal{L}_1 intersect transversally at finitely many points. We call this assumption *finite intersection* of the Lagrangians. It holds if one of the Lagrangians is compact. More generally it holds if one of the correspondences $L_{i,i+1}$ is compact and all the others are proper in the following sense.

Definition 3.3.1. A correspondence $L \subset M^- \times M'$ is called *proper* if for each point $y \in M'$ the set $\{x \in M \mid (x, y) \in L\}$ is compact.

We denote the isotoped Lagrangians by the same notation. Let J_i be a compatible almost complex structure on M_i . If M_i is Stein then we require J_i to be invariant under the flow of Liouville vector field outside a compact set. We call such an almost complex structure *asymptotically invariant*. From the J_i we get an almost complex structure

$$J = (-J_0, J_1, \dots, (-1)^{n-1} J_n)$$

on M . Let $x, y \in \mathcal{L}_0 \cap \mathcal{L}_1$ and let J_t be a one parameter family of almost complex structures on M for which there is an $r_0 > 0$ such that $J_t = J$ outside $M_{\leq r_0}$ for all t . Denote by $\mathcal{M}(x, y)$ the moduli space of the strips

$$u : \mathbb{R} \times [0, 1] \rightarrow \mathcal{Y}$$

such that $\frac{\partial}{\partial t} u(s, t) = J_t \frac{\partial}{\partial s} u(s, t)$ and $u(s, i) \in \mathcal{L}_i$ for $i = 0, 1$ and $u(-\infty, t) = x, u(\infty, t) = y$. \mathbb{R} acts on $\mathcal{M}(x, y)$ by shifting the parametrization.

Consider the abelian group $CF(\mathcal{L}_0, \mathcal{L}_1)$ generated freely by the intersection points of the two Lagrangians. The following differential makes $CF(\mathcal{L}_0, \mathcal{L}_1)$ into a chain complex and the Floer cohomology $HF(\mathcal{L}_0, \mathcal{L}_1)$ is defined to be the cohomology of this complex.

$$(28) \quad \partial x = \sum_{y \in \mathcal{L}_0 \cap \mathcal{L}_1} \# \mathcal{M}_1(x, y) / \mathbb{R}$$

Here $\mathcal{M}_1(x, y)$ is the one dimensional part of the moduli space. The count is a priori in $\mathbb{Z}/2$. To be able to define Floer cohomology groups as \mathbb{Z} one needs coherent orientations on the moduli spaces cf. [4]. For this invariant to be well-defined one has to take care of the following issues: transversality of moduli spaces, nonexistence of bubbling, compactness and invariance under Hamiltonian isotopy. In the following discussion we assume that the first two criteria hold and focus on the last two. First compactness.

Lemma 3.3.2. *Assume (M, ψ) is a Stein manifold, $\mathcal{L}_0, \mathcal{L}_1$ are C -allowable Lagrangians for some $C \geq r_0$ and that $M_{\leq C}$ contains the intersection points of the Lagrangians. Then for any Riemann surface S with boundary and any J -holomorphic map $u : S \rightarrow M$ with $u(\partial S) \subset \mathcal{L}_0 \cup \mathcal{L}_1$, the image of u lies in $M_{\leq C}$ (which is independent of S and u).*

Proof. [3], [14] For $u \in \mathcal{M}(x, y)$, $\psi \circ u$ cannot have a maximum on the interior of the curve outside $M_{\leq C}$ by the maximum principle. This is because u is J holomorphic outside $M_{\leq r_0}$. Assume it has a maximum on a boundary point p ie. $\max \psi \circ u = \psi \circ u(p) = R > C$. We can pick holomorphic coordinates on S in a neighborhood of p and therefore regard u , in a neighborhood of $p = (s_0, 1)$, as a function on some rectangle $[l_1, l_2] \times [1 - \delta, 1]$. We have $d\psi(\frac{\partial}{\partial s}u)(s_0, 1) = 0$. So $\frac{\partial}{\partial s}u(s_0, 1) \in T\mathcal{L}_1 \cap TM_R$. By assumption the intersection is transverse therefore it is Legendrian so $d\psi(J\frac{\partial}{\partial s}u(s_0, 1)) = 0$. But we have $\frac{\partial}{\partial t}u = J\frac{\partial}{\partial s}u$ so $d\psi(\frac{\partial}{\partial t}u)(s_0, 1) = 0$. This contradicts the strong maximum principle which implies $d\psi(\frac{\partial}{\partial t}u) > 0$. \square

This enables us to apply the rescaling argument to show that the limit of a bounded energy sequence of such curves is either a broken trajectory or a curve with sphere or disc bubbles. Therefore we can compactify $\mathcal{M}(x, y)$ by adding these limiting curves to it. If in addition both M and the Lagrangians are exact, no bubbling occurs and so the sum in (28) is finite and we get $\partial^2 = 0$.

An important property of Floer cohomology in compact manifolds is invariance under Hamiltonian isotopy. However this invariance does not

hold for isotopies with noncompact support. We have the following theorem of Oh.

Theorem 3.3.3 (Oh [14]). *Let M be a Stein manifold and L, L' Lagrangian submanifolds of M , H_t , $t \in [0, 1]$, a time-dependent conical Hamiltonian on M and L_t the image of L under time t map of H . Assume $\cup_t L_t \cap L'$ is compact.*

- i) We can find another Hamiltonian H' such that the time one map of H' equals that of H and H' is positive on $\cup_{s \in [\delta, 1-\delta]} L \cap L_s$ for some $0 < \delta < 1$.*
- ii) If in addition H' is conical then there is a canonical isomorphism $HF(L_0, L') \cong HF(L_1, L')$.*

Proposition 3.3.4. *Let L, L' be two Lagrangians in a Stein manifold (M, ψ) which are C -allowable and satisfy the finite intersection condition. Then the Floer cohomology $HF(\mathfrak{C}_c(L), \mathfrak{C}_c(L'))$ is well-defined and is invariant under conical Hamiltonian isotopies. It is independent of c when $c > C$ and $L \cap L' \subset M_{\leq c}$.*

Proof. It follows from the last two results along with those in section 3.1. \square

In order for Floer cohomology to define a well-defined map on the symplectic category, we must understand the effect of composition of Lagrangian correspondences on Floer cohomology. The following important *Functoriality Theorem* is proved in [19].

Theorem 3.3.5 (Wehrheim, Woodward [19]). *Let $\underline{L} = (L_{0,1}, L_{1,2}, \dots, L_{n-1,n})$ be a generalized Lagrangian correspondence between manifolds M_0, \dots, M_n such that for some $0 < j < n$ the composition $L_{j,j+1} \circ L_{j-1,j}$ is embedded. Denote*

$$\underline{L}' = (L_0, \dots, L_{j,j+1} \circ L_{j-1,j}, \dots, L_n).$$

Assume the M_i are compact and monotone with the same monotonicity constant and all $L_{i,i+1}$ as well as (25) and (26) are monotone. Assume in addition that each $L_{i,i+1}$ is oriented, relatively spin, graded and its minimal Maslov number is at least three. Then with the induced grading and relative spin structure on $L_{i,i+1} \circ L_{i-1,i}$ there is a canonical isomorphism

$$(29) \quad HF(\underline{L}) \cong HF(\underline{L}')$$

of graded abelian groups.

Proposition 3.3.6. *If the M_i are Stein and $L_{i,i+1}$ are allowable, relatively spin and satisfy the finite intersection condition then (29) holds.*

Proof. It is easy to see that the generators for the two Floer groups are in one-to-one correspondence. Take $x, y \in \mathcal{L}_0 \cap \mathcal{L}_1$ and let $\mathcal{M}_\delta(x, y)$ be the moduli space of pseudoholomorphic strips $u = (u_0, \dots, u_n)$ where

$$u_i : [0, 1] \times \mathbb{R} \rightarrow M_i$$

if $i \neq j$ and $u_i : [0, \delta] \rightarrow M_j$ and with boundary condition for u given by \underline{L} and with x and y as asymptotic points. Let $\mathcal{M}'(x, y)$ be the moduli space of strips

$$(u_0, \dots, u_{i-1}, u_{i+1}, \dots, u_n) : [0, 1] \times \mathbb{R} \rightarrow M_0 \times \dots \times M_n$$

with boundary condition \underline{L}' and asymptotic points x, y . Since the M_i are Stein and the Lagrangians are allowable and satisfy the finite intersection property then Lemma 3.3.2 implies that the holomorphic curves in $\mathcal{M}_\delta(x, y)$ and $\mathcal{M}'(x, y)$ stay in a compact submanifold of $M_0 \times \dots \times M_n$ (which doesn't depend on δ) and so the proof proceeds as in [19] to show that for δ small enough the two moduli spaces are bijective. Note that exactness of the $L_{i,i+1}$ implies monotonicity and rules out bubbling so we do not need the assumption on the minimal Maslov index in 3.3.5. \square

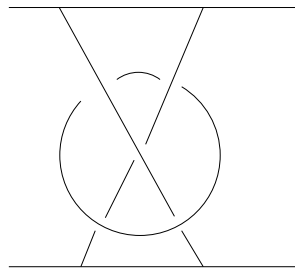
Remark 3.3.7. Assume we have two Hamiltonian isotopic Lagrangian correspondences L_0, L_1 and another correspondence L' such that both compositions $L_0 \circ L'$ and $L_1 \circ L'$ are embedded. In general we don't know if $L_0 \circ L'$ and $L_1 \circ L'$ are Hamiltonian isotopic or not. However the proof of the Functoriality Theorem implies that Floer cohomology is invariant under such an isotopy.

4. A SYMPLECTIC VALUED TOPOLOGICAL FIELD THEORY

4.1. Tangles. A tangle T is defined to be a compact one-dimensional submanifold of (a diffeomorphic image of) $\mathbb{C} \times [0, 1]$ such that $i(T) := (T \cap \mathbb{C}) \times \{0\} \subset \mathbb{R} \times \{0\}$ and $t(T) := (\partial T \cap \mathbb{C}) \times \{1\} \subset \mathbb{R} \times \{1\}$. The second assumption makes $i(T)$ and $t(T)$ ordered sets. In this paper we deal only with tangles with an even number of initial points and end points. If $\#i(T) = 2m, \#t(T) = 2n$ we say T is an (m, n) -tangle and write mTn . We also allow m and/or n to be zero.

Definition 4.1.1. Two tangles T, T' are called *equivalent* if there is a continuous family T_t of tangles for $t \in [0, 1]$ such that $T_0 = T$ and $T_1 = T'$ and the order of $i(T_t)$ and of $t(T_t)$ is fixed.

Two tangles T_1, T_2 can be composed (concatenated) if $t(T_1) = i(T_2)$. Two equivalence classes $[T_1]$ and $[T_2]$ of tangles can be composed if $\#t(T_1) = \#i(T_2)$ and composition is done using the ordering on $t(T_1)$ and $i(T_2)$. Composition of tangles is denoted by juxtaposition. We will use the notation

FIGURE 3. A $(1,1)$ -tangle

id_m , $\cap_{i;m}$, $\cup_{i;m}$ and $\sigma_{i;m}$ for the elementary tangles in figures 4 and 5 where m denotes the number of the strands. We might ignore m when there is no confusion. When we say a tangle T is equivalent to, say, $\cap_{i;m}$, we implicitly have a one to one correspondence between $i(T)$ and $\{1, 2, \dots, 2m-2\}$ and also between $t(T)$ and $\{1, 2, \dots, 2m\}$ in mind.

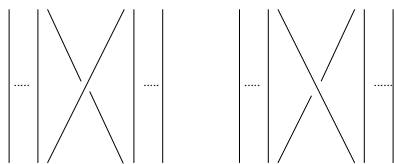
A *decomposition* of T is a sequence of tangles

$$(30) \quad n_0 T_1 n_1 T_2 \dots n_{l-1} T_l n_l \quad n_0 = m, n_l = n$$

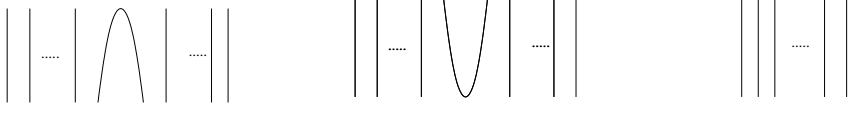
such that T is equivalent to $T_1 T_2 \dots T_l$. It is possible to express any T (not uniquely) as a composition of elementary tangles. *Crossingless matchings* (section 2.5) are a special class of $(0, n)$ or $(n, 0)$ -tangles. Given a set of $2n$ points on the plane, a crossingless matching is a set of n nonintersecting curves each joining a pair of the given points. In the context of tangles a crossingless matching is regarded as a subset of $\mathbb{C} \times [0, 1]$.

Definition 4.1.2. Let c_n be the set of isotopy (in \mathbb{C}) classes of crossingless matchings between $2n$ points.

The cardinality of c_n equals the n th Catalan number.

FIGURE 4. The braids σ_i and σ_i^t

One can define a category **Tang** whose objects are natural numbers and $\text{hom}(m, n)$ consists of equivalence classes of (m, n) -tangles. **Tang** has

FIGURE 5. $\cap_{i,m}$, $\cup_{i,m}$ and id_m

a monoidal structure given by putting two tangles kTl and mTn “side-by-side” and obtain a $(k+m, l+n)$ -tangle. We denote this by $T \oplus T'$. To each (m, n) -tangle T there is assigned a “mirror image” T^t which is a (n, m) -tangle. There is a generators and relations description of **Tang** due to Yetter [21] whose proof relies on Reidemeister’s description of plane diagram moves.

Lemma 4.1.3 (Yetter [21]). *The following are all the commutation relations between elementary tangles where “=” means equivalence. If $|i-j| > 1$ we have:*

$$(31) \quad \sigma_i \sigma_j = \sigma_j \sigma_i$$

$$(32) \quad \cap_{i;m} \cup_{j;m} = \cup_{j-2;m} \cap_{i;m}$$

$$(33) \quad \cap_{i;m} \sigma_{j;m} = \sigma_{j-2;m} \cap_{i;m} \quad \cup_{i;m} \sigma_{j;m} = \sigma_{j+2;m} \cup_{i;m}.$$

And for any i we have:

$$(34) \quad \sigma_i \cup_i = \cup_i$$

$$(35) \quad \sigma_i \sigma_i^t = id$$

$$(36) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$(37) \quad \cap_{i;m} \cup_{i+1;m} = id_{m-1}$$

$$(38) \quad \sigma_i \cup_{i+1} = \sigma_{i+1}^t \cup_i \quad \sigma_i^t \cup_{i+1} = \sigma_{i+1} \cup_i.$$

Given two decompositions of two equivalent tangles, the following lemma provides a natural way of converting one to the other.

Lemma 4.1.4. *If T, T' are two equivalent tangles and $D : T = T_1 T_2 \cdots T_m$, $D' : T' = T'_1 T'_2 \cdots T'_m$ are decompositions into elementary tangles then one decomposition can be converted to the other by a sequence of the above moves.*

Proof. We can regard each decomposition as being inside $\mathbb{C} \times [0, 1]$. We can also find Morse functions f, f' on T and T' respectively such that the decomposition of T (resp T') by critical sets of f (resp. f') yields D (resp.

D'); in other words f has critical points separating each T_i from T_{i+1} and no more. There is a diffeomorphism ϕ of $\mathbb{C} \times [0, 1]$ such that $\phi(T) = T'$. The reason is that because T and T' are equivalent, there is a family $T(t)$ of tangles with the same boundary points such that $T(0) = T, T(1) = T'$. Because strands of $T(t)$ never intersect, we can obtain a time dependent vector field on $T(t)$ by differentiation. This vector field can be smoothly extended to $\mathbb{C} \times [0, 1]$. Let ϕ be the time one map of this vector field. Then $\phi(T) = T'$.

Let $f'' = \phi^* f'$ and $T''_i = \phi^{-1}(T'_i)$. So we get a decomposition $D'' : T''_1, \dots, T''_m$ of T induced by f'' . According to Cerf theory [2], there is a family f_t of smooth functions such that $f_0 = f, f_1 = f''$ and f_t is Morse except for finitely many times t_1, \dots, t_k and at each t_i a pair of canceling critical points is introduced or deleted. Since T is one dimensional, each t_i has the effect of either (1) merging two adjacent handles (i.e. two adjacent elementary tangles) or decomposing one into two or (2) the effect of the move (37) above. In the first case the effect of merging two adjacent handles and then separating into two new ones is equivalent to one of the above moves.

Now between each t_i, t_{i+1} , each handle can be isotoped to an equivalent one. The only way this can change the decomposition is by changing the value of f_t at critical points and thereby changing the order of the critical points in the decomposition. This also has the effect of commuting the handlebodies in the decomposition.

□

The invariant that we define in this paper is an invariant of *oriented tangles*. An oriented tangle comes with an orientation of each one of its components. Two example are shown in Figure 6. When considering commutation relations between tangles we ignore the orientation.

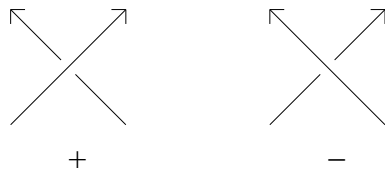


FIGURE 6. Oriented braids σ^+ and σ^-

4.2. The functor. Let

$$(39) \quad n_0 T_1 n_1 T_2 \dots n_{l-1} T_l n_l$$

be a decomposition of an oriented tangle T . Set $\nu_j = i(T_j)$ and $\nu_{l+1} = t(T_l)$. We have $\nu_i \in \text{Conf}_{n_i}$ for $i = 0, \dots, l$. To each T_i we want to associate a Lagrangian correspondence $L_{i,i+1} = L_{T_i}$ between \mathcal{Y}_{n_i, ν_i} and $\mathcal{Y}_{n_{i+1}, \nu_{i+1}}$. In this way we can associate to T a generalized Lagrangian correspondence

$$(40) \quad \Phi(T) = (L_{0,1}, L_{1,2}, \dots, L_{n-1,n})\{-m - w\}$$

from \mathcal{Y}_n to \mathcal{Y}_m . Here m and w are the number of cups and the writhe (number of positive crossings minus the number of negative ones) of the decomposition respectively.

If T_k is an elementary braid in Br_{2m} , we set L_{T_k} to be $\text{graph}(h_\beta^{res})$ regardless of the orientation of the braid. Of course we can extend this definition to any braid. We note that the symplectomorphisms h_β^{res} are defined only on compact submanifolds of the \mathcal{Y}_m however this does not cause any problem since we are going to take the \mathfrak{C}_c of these Lagrangians. Let V_i be the relative vanishing cycle for the map f in Lemma 2.1.3 where i th and $(i+1)$ th eigenvalues (μ_i, μ_{i+1}) of ν_k come together at some point μ . There is a theorem of T. Perutz (generalizing a former result of P. Seidel) which describes monodromy maps around singularities of symplectic Morse-Bott fibrations as Dehn twists. Recall that a symplectic Morse-Bott fibration (also called Lefschetz-Bott fibrations) over a disc D consists of an almost complex manifold (E, J) , a closed two-form Ω on E and a J -holomorphic map $\pi : E \rightarrow D$ such that the critical point set of π is a submanifold of E and the complex Hessian matrix of π is nondegenerate. The form Ω is required to be a symplectic form when restricted to each fiber.

Theorem 4.2.1 (Perutz [15], Theorem 2.19). *Let $\pi : E \rightarrow D$ be a symplectic Morse-Bott fibration over the closed disc D which has only the origin as singular value. If in addition the fibration is normally Kahler, ie. a neighborhood of the critical point set of π is foliated by J -complex normal slices such that J restricted to each slice is integrable and the restriction of Ω to each fiber is Kahler, then the monodromy map around the origin is Hamiltonian isotopic to the fibred Dehn twist τ_V along the vanishing cycle V for the map π .*

Therefore using the local picture of the Lemma 2.1.3 we see that if we have a subset $B \subset \mathcal{Y}_m$ for which the naive (non-rescaled) parallel transport map $h_{\sigma_i}|_B$ is well-defined then

$$(41) \quad h_{\sigma_i} \cong \tau_{V_i}.$$

The reason is that since the naive parallel transport is well-defined for all points of B , we can shrink the rescaling parameter in (12) to zero and thereby isotope $h_{\sigma_i}^{res}$ to h_{σ_i} .

Lemma 4.2.2. *Let $\nu = \{\mu_1, \dots, \mu_{2m}\} \in \text{Conf}_{2m}^0$ and $\gamma : [0, 1] \rightarrow \mathfrak{h}/W$ such that $\gamma(0) = \nu$ and as $s \rightarrow 1$, μ_{2i-1} and μ_{2i} approach each other linearly and collide at some μ . Then $h_{\sigma_i}^{res}$ restricted to $\mathcal{S}_m^{sub, \mu}$ is identity. Similarly if $\mu_{2k-1}, \mu_{2k}, \mu_{2k+1}$ come together at some μ then σ_i and σ_{i+1} act trivially on $\mathcal{S}_m^{sub3, \mu}$.*

Proof. We use the picture of Lemma 2.1.5. Both statements follow from (41) and the fact that fibred Dehn twists are identity outside a small neighborhood of the spherical fibres. \square

Let V_{ix} denote the (S^2) fiber of V_i over x . We grade τ_{V_i} in such a way that

$$(42) \quad \tau_{V_i} V_{ix} = V_{ix}\{1\}$$

and the grading function vanishes outside a neighborhood of V_i . This grading is unique. (Lemma 5.6 in [16])

Remark 4.2.3. Monodromy actions of braid group on symplectic manifolds were first constructed in [11]. Parallel transport maps h_β^{res} form a homomorphism from $\pi_1(\text{Conf}_{2m}) = Br_{2m}$ into the π_0 of $\text{Sympl}(\mathcal{Y}_m)$. In particular symplectomorphisms associated to elementary braids satisfy Artin's commutation relations. (Symplectic manifolds used in [11] are compact with boundary. The manifolds \mathcal{Y}_m in [17] that we use here can be obtained from them by attaching an infinite cone.)

If $T_i = \cup_{j;m}$, we define a Lagrangian $L_{\cup_{k;m}}$, regardless of the orientation of $\cup_{k;m}$, as follows. The result depends on a real parameter $R > 0$. To simplify the notation we set $k = 2j - 1, l = 2j$.

With ν_i as given above let $\nu = \nu_i = \{\mu_1, \dots, \mu_{2m}\}$. Let γ be a curve in Conf_{2m}^0 such that $\gamma(0) = \nu_i$ and as $s \rightarrow 1$, μ_k and μ_l approach each other linearly and collide at a point μ' . For example we can take

$$\gamma(t) = \{\mu_1, \dots, \mu_k + t(\mu_l - \mu_k)/2, \dots, \mu_l - t(\mu_l - \mu_k)/2, \dots, \mu_{2m}\}$$

provided that $\mu_k + t(\mu_l - \mu_k)/2$ does not intersect the other μ_i . Set $\nu^{k,l} = \nu \setminus \{\mu_k, \mu_l\}$, $\nu' = \gamma(1)$. We use Lemma 2.1.3 to identify a neighborhood of $\mathcal{S}_m^{sub, \mu'}$ in \mathcal{S}_m locally with $\mathcal{S}_m^{sub, \mu'} \times \mathbb{C}^3$. This induces a Kahler form and hence a metric on $\mathcal{S}_m^{sub, \mu'} \times \mathbb{C}^3$. We perturb the complex structure outside a compact ball of radius ρ (to be chosen below) so that outside that set the

resulting metric equals the product metric. Now we use the relative vanishing cycle construction for the whole $\mathcal{S}_m^{sub, \mu'}$. It yields (after restriction) a sphere bundle $V = V_{\gamma(1-s)}(\mathcal{S}_m^{sub, \mu'}) \subset \mathcal{Y}_{m, \gamma(1-s)}$ for small s with projection $\pi : V \rightarrow \mathcal{Y}_{m, \nu'} \cap O^{sub, \mu'_k}$. This construction works because the metric equals the product metric outside a compact set.

We denote the image of V under parallel transport map along $-\gamma$, i.e. $h_{\gamma|_{[0, 1-s]}}^{-1}(V) \subset \mathcal{Y}_{m, \nu}$, by the same notation V . Composing π with the parallel transport map $h_{\gamma|_{[0, 1-s]}}^{-1}$ we obtain a projection $\pi : V \rightarrow \mathcal{Y}_{m, \nu'} \cap O^{sub, \mu'}$ which is a S^2 bundle. By Lemma 2.1.3, $\mathcal{Y}_{m, \nu'} \cap O^{sub, \mu'}$ can be identified with $\mathcal{D}_{m-1, \nu'}$ from (13). Let δ be a geodesic in Conf_{2m}^0 joining ν' to $\nu^{k, l}$. We can use parallel transport map (14) along the curve δ to map $\mathcal{D}_{m-1, \nu'}$ to $\mathcal{D}_{m-1, \nu^{k, l} \cup \{0, 0\}}$. The latter can be identified with $\mathcal{Y}_{m-1, \nu^{k, l}}$. So we obtain a fibration $\pi : V \rightarrow \mathcal{Y}_{m-1, \nu^{k, l}}$. We can use this map π to pull V back to $\mathcal{Y}_{m-1, \nu^{k, l}} \times \mathcal{Y}_{m-1, \nu^{k, l}}$. Let $\cup_{j; m}$ be its restriction to the diagonal. It is a Lagrangian submanifold of $\mathcal{Y}_{m, \nu_i}^- \times \mathcal{Y}_{m-1, \nu_i^{k, l}} = \mathcal{Y}_{m, \nu_i}^- \times \mathcal{Y}_{m-1, \nu_{i+1}}$. Let $\psi = \psi_1 + \psi_2$ be the plurisubharmonic function on $\mathcal{Y}_{m, \nu_i}^- \times \mathcal{Y}_{m-1, \nu_{i+1}}$. We can choose ρ in such a way that the inverse image of $\psi = R$ lies inside the ball of radius ρ . We have a projection $\pi : L_{\cup_{j; m}} \rightarrow \Delta \subset \mathcal{Y}_{m-1, \nu_{i+1}}^- \times \mathcal{Y}_{m-1, \nu_{i+1}}$.

As in the case of Lagrangians from crossingless matchings, replacing the curve γ with another curve in the same homotopy class (with fixed endpoints) results in a new $L_{\cup_{j; m}}$ which is Lagrangian isotopic to the former one. Since the first homology group of this Lagrangian is zero, this isotopy is exact.

We grade $L_{\cup_{i; m}}$ as follows. Lemma 4.2.7 below tells us that fibers of L_{\cup_i} and $L_{\cup_{i+1}}$ over each point of the diagonal intersect transversely at only one point. We choose the grading in such a way that the absolute Maslov index of this intersection point (with regard to the two S^2 fibres) equals one. We can use Lemma 2.1.3 and isotope L_{\cup_i} to $\Delta \times \sqrt{z}S^2$ for some small $z \in \mathbb{C}$. One can see by direct computation that $\alpha_{L_C} = \alpha_{\Delta} \cdot \frac{z}{|z|}$. The function α_{Δ} is constant. This means that the grading on L_{\cup_i} is determined by the choice of a branch of $\arg(z)$. In particular such a grading is a constant function:

$$(43) \quad \tilde{L}_{\cup_i} = c_i.$$

We choose this c_i to be the same for every i . This together with the formula (18) (for $n = 2$) imply that the absolute Maslov index of each fiberwise intersection point in $L_{\cup_i} \cap L_{\cup_{i+1}}$ equals 1. Construction for $\cap_{k; m}$ is similar.

The following lemma insures that the Lagrangian we assign to crossingless matching agrees with that of Seidel and Smith.

Lemma 4.2.4. *If $C \in \mathcal{C}_m$ is a crossingless matching and arcs of C are isotopic to $T_{i_1, j_1}, \dots, T_{i_m, j_m}$ where each T is a cap or a cup then L_C is isotopic to $L_{T_{i_1, j_1}} \circ \dots \circ L_{T_{i_m, j_m}}$.*

Proof. We use induction on m . The base case is vacuous. For the induction step we note that our construction is the same as the induction step in the construction of Seidel and Smith (section 2.5 above) except for the base of the fibration. \square

In order for Φ to define a functor, we must verify that the above correspondences satisfy the same commutation relations as the tangles they are associated to. First we have the following cf. remark 4.2.3.

Lemma 4.2.5. *We have $L_{\sigma_i} L_{\sigma_{i+1}} L_{\sigma_i} = L_{\sigma_{i+1}} L_{\sigma_i} L_{\sigma_{i+1}}$ and if $|i - j| > 1$, $L_{\sigma_i} L_{\sigma_j} = L_{\sigma_j} L_{\sigma_i}$.*

Lemma 4.2.6. *We have*

$$(44) \quad L_{\cap_{i,m}} L_{\sigma_j} \simeq L_{\sigma_{j-2}} L_{\cap_{i,m}} \quad L_{\cup_{i,m}} L_{\sigma_j} \simeq L_{\sigma_{j+2}} L_{\cup_{i,m}}$$

if $|i - j| > 1$ and for any i we have:

$$(45) \quad L_{\cap_{i,m}} L_{\sigma_i} \simeq L_{\cap_{i,m}} \{1\} \quad L_{\sigma_i} L_{\cup_{i,m}} \simeq L_{\cup_{i,m}} \{1\}$$

$$(46) \quad L_{\sigma_i} L_{\cup_{i+1}} \simeq L_{\sigma_{i+1}}^t L_{\cup_i} \quad L_{\sigma_{i+1}} L_{\cup_i} \simeq L_{\sigma_i}^t L_{\cup_{i+1}}$$

where “ \simeq ” means exact isotopy.

Proof. Let $\beta : [0, 1] \rightarrow \text{Conf}_{2m}^0$ be a braid such that $\beta(0) = \beta(1) = \nu$. Note that in the construction of L_{\cap} if we replace the curve γ with $\beta * \gamma$ and also replace δ with $\delta * \alpha$ where α joins $\nu^{(k)}$ to $\nu \setminus \{\beta(2k-1), \beta(2k)\}$, the construction will yield $h_{\beta}(\cap_{k;m})$. In general the basepoint $\nu^{(k)}$ or $\nu \setminus \{\beta(2k-1), \beta(2k)\}$ is of no importance so we can assume α to be constant. If β equals σ_k then $\beta * \gamma$ joins μ_{2k-1}, μ_{2k} and fixes the other eigenvalues so the construction will yield the same L_{\cap_k} . If $\beta = \sigma_{k+1}$ then $\beta * \gamma$ the result will be the same as starting with \cap_{i+1} and using $\beta = \sigma_k$. These two facts imply the above isotopies ignoring grading. As another proof which shows equality of graded Lagrangians, we appeal to (42) which implies (45). To prove (46), we use Lemma 5.8 in [16] which asserts if L_0, L_1 are two graded Lagrangian spheres whose intersection consists of only one point of Maslov index one and with the grading of the Dehn twists and chosen as above, we have

$$(47) \quad \tau_{L_0} L_1 = \tau_{L_1}^{-1} L_0$$

as graded Lagrangians. Now we use Lemma 2.1.5 to translate the picture into that of section 2.3. So we can identify L_{\cup_i} with $\Lambda_{\alpha_1} \times_{S^1} \Delta$ and $L_{\cup_{i+1}}$ with $\Lambda_{\alpha_2} \times_{S^1} \Delta$. Because h_{σ_i} and $h_{\sigma_{i+1}}$ act trivially on Δ by 4.2.2, we can identify them with Dehn twists around Λ_{α_1} and Λ_{α_2} respectively. Our choice of grading (43) for L_{\cup_i} implies that the hypothesis of (47) are met. This immediately implies (46). Note that because Λ_{α_i} are two dimensional, changing the order of them does not change the Maslov index of the intersection point. \square

Lemma 4.2.7. *We have the following commutation relations where “ \simeq ” means exact isotopy.*

$$(48) \quad L_{\cap_{i,m}} L_{\cup_{j,m}} \simeq L_{\cup_{j-2,m-2}} L_{\cap_{i,m-2}} \quad \text{if } |i-j| > 1.$$

$$(49) \quad L_{\cap_{i,m}} L_{\cup_{i+1,m}} \simeq L_{id_{m-1}} \{-1\} \quad \text{for any } i.$$

Proof. To prove (48) let $\nu = \{\mu_1, \dots, \mu_{2m}\}$ and ν', ν'', ν''' be ν minus $\{\mu_{2i-1}, \mu_{2i}\}$, $\{\mu_{2j-1}, \mu_{2j}\}$, $\{\mu_{2i-1}, \mu_{2i}, \mu_{2j-1}, \mu_{2j}\}$ respectively. Therefore

$$\begin{aligned} L_{\cap_{i,k}} &\subset \mathcal{Y}_{m-1,\nu'} \times \mathcal{Y}_{m,\nu} & \text{and} & \quad L_{\cup_{j,m}} \subset \mathcal{Y}_{m,\nu} \times \mathcal{Y}_{m-1,\nu''}. \\ L_{\cup_{j-2,k-2}} &\subset \mathcal{Y}_{m-1,\nu'} \times \mathcal{Y}_{m-2,\nu'''} & \text{and} & \quad L_{\cup_{i,m-2}} \subset \mathcal{Y}_{m-2,\nu'''} \times \mathcal{Y}_{m-1,\nu''}. \end{aligned}$$

We have projections $\pi_i : L_{\cap_{i,m}} \rightarrow \mathcal{Y}_{m-1,\nu'}$, $\pi_j : L_{\cup_{j,m}} \rightarrow \mathcal{Y}_{m-1,\nu''}$ given in the construction of these Lagrangians. We have

$$\begin{aligned} L_{\cap_{i,m}} L_{\cup_{j,m}} &= \{(m_1, m_2) | \exists m, (m_1, m) \in L_{\cap_{i,k}}, (m, m_2) \in L_{\cup_{j,k}}\} = \\ &= \{(m_1, m_2) | \exists m, m_1 = \pi_i(m), m_2 = \pi_j(m)\}. \\ L_{\cup_{j-2,m-2}} L_{\cap_{i,m-2}} &= \{(m_1, m_2) | \exists m, (m_1, m) \in L_{\cup_{j-2,k-2}}, (m, m_2) \in L_{\cap_{i,k-2}}\} = \\ &= \{(m_1, m_2) | \exists m, \pi_j(m_1) = m, \pi_i(m_2) = m\} \\ &= \{(m_1, m_2) | \pi_j(m_1) = \pi_i(m_2)\} \end{aligned}$$

By the remark after the Lemma 2.1.3, these projections are given by projection to the first and second factor in $\mathfrak{sl}(E_{\mu_1}) \oplus \mathfrak{sl}(E_{\mu_2}) \oplus \zeta$ and so $\pi_i \pi_j = \pi_j \pi_i$ from which the equality of the compositions follows.

As for (49) we must show that $L_{\cap_i} \circ L_{\cup_{i+1}}$ equals the diagonal $\Delta \subset \mathcal{Y}_m \times \mathcal{Y}_m$. Using Lemma 4.2.6 we see that

$$h_{\sigma_i \sigma_{i+1}} L_{\cap_i} = L_{\cap_{i+1}}.$$

So we are reduced to showing that $L_{\cap_i} \cap h_{\sigma_i \sigma_{i+1}}(L_{\cap_i}) \simeq \Delta$. Let $\nu = (\mu_1, \dots, \mu_{2m+2})$ be such that $\mu_i + \mu_{i+1} + \mu_{i+2} = 0$ and

$\bar{\nu} = (\mu_1, \dots, \mu_{i-1}, 0, \mu_{i+3}, \dots, \mu_{2m+2})$ also $\hat{\nu} = (\mu_1, \dots, \mu_{i-1}, 0, 0, 0, \mu_{i+3}, \dots, \mu_{2m+2})$. We take $\mathcal{Y}_{m+1} = \mathcal{Y}_{m+1, \nu}$ and $\mathcal{Y}_m = \mathcal{Y}_{m, \bar{\nu}}$. The latter can be identified with $\mathcal{D}_{m+1, \hat{\nu}}$ which is $(\chi^{-1}(\hat{\nu}) \cap \mathcal{S}_{m+1})^{sub3,0}$. Lemma 2.1.5 gives us a \mathbb{C}^4 -bundle \mathcal{L} over $\mathcal{S}_{m+1}^{sub3,0}$ with a local symplectomorphism ϕ from \mathcal{L} to a neighborhood U of $\mathcal{S}_{m+1}^{sub3,0}$ in \mathcal{S}_{m+1} . We identify $\mathcal{Y}_{m+1, \nu}$ with its image under ϕ and so $\Delta \subset \mathcal{L} \times \mathcal{S}_{m+1}^{sub3,0}$. $\mathcal{Y}_{m, \bar{\nu}}$ lies in the zero section and L_{\cap_i} is the pullback of a sphere bundle over the zero section to Δ . Lemma 4.2.2 tells us that $h_{\sigma_i \sigma_{i+1}}$ is identity when restricted to Δ therefore $h_{\sigma_i \sigma_{i+1}}(L_{\cap_i})$ is another sphere bundle over Δ . We show that fibers of this two bundles over each point of Δ intersect at only one point so $L_{\cap_i} \cap h_{\sigma_i \sigma_{i+1}}(L_{\cap_i})$ is Lagrangian isotopic to Δ . Since these Lagrangians have vanishing first cohomology groups this isotopy is exact.

Let L_1 and L_2 be fibers of L_{\cap_i} and $h_{\sigma_i \sigma_{i+1}}(L_{\cap_i})$ over a point of Δ . So

$$(50) \quad L_2 = h_{\sigma_i \sigma_{i+1}}(L_1).$$

By Lemma 2.3.1, L_{\cap_i} is isotopic to $\Lambda_{d, \alpha_1} = \Delta \times_{S^1} \Lambda_{\alpha}$ so L_1 can be identified with Λ_{α_1} . When we perform $h_{\sigma_i \sigma_{i+2}}$, the leftmost and the rightmost zeros in Figure 1 (which are the “fiberwise” eigenvalues) get swapped; therefore \mathbf{c} is sent to \mathbf{c}' and so $L_2 = \Lambda_{\mathbf{c}}$. Since $\Lambda_{\mathbf{c}} = \bigcup_{r=0}^1 C_{d, z, \mathbf{c}(r)}$ and $C_{d, z, a}$ is given by $C_{d, z, a} = \{(b, c) : |b| = |c|, a^3 - da - z = -bc\}$, we see that since the curves \mathbf{c} and \mathbf{c}'' intersect at only one point (i.e. the first root of $a^3 - ad - z = 0$) then $L_1 = \Lambda_{\alpha_1}$ and $L_2 = \Lambda_{\alpha_2}$ intersect at only one point so we are done.

We chose the gradings (43) for L_{\cap_i} to be a constant function c and be the same for all i . Let S_{ix} be the fiber of L_{\cap_i} over x . Then the fiber of $L_{\cap_{i+m}} \circ L_{\cup_{i+1, m}}$ over x equals $S_{ix} \circ S_{i+1x}^t$. Grading of S_{i+1x}^t equal $1 - \tilde{S}_{i+1x}$ by (24). Therefore the grading of the point $S_{ix} \circ S_{i+1x}^t$ is $c + 1 - c = 1$ which implies (49). \square

Theorem 4.2.8. *The assignment Φ in (40) gives a functor from the category of tangles to the symplectic category so it defines a graded symplectic valued topological field theory.*

Proof. Follows from 4.1.4 and 4.2.5 to 4.2.7. We see that difference in grading for each commutation relation gets canceled by the change in the writhe plus number of cups. The only commutation relations which involve grading shift are (45) and (49) which happen to be the only ones involving change in $-m - w$. For (49), $-m$ plus the degree shift is equal on both sides of the equation. Note that if T contains $\cap_i(\sigma_i^\epsilon)^\pm$ where $\epsilon = 1, -1$ and \pm is the

writhe of σ_i then $\pm\epsilon$ has to be equal to -1 . This implies that $-w$ plus the grading shift is equal on both sides of (45). \square

4.3. The invariant. We can obtain tangle invariants at two levels from the symplectic valued topological field theory Φ : a functor valued invariant and a group valued one. According to [13] the *generalized Fukaya category* $\text{Fuk}^\#(M)$ of a symplectic manifold M is an A_∞ category whose objects are compact generalized Lagrangian correspondences between a point and M and morphisms between two such objects \underline{L}_0 and \underline{L}_1 are the elements of Floer chain complex for $\underline{L}_0^t \# \underline{L}_1$. The A_∞ structure on $\text{Fuk}^\#(M)$ is given by counting holomorphic “quilted polygons”. More precisely the maps

$$(51) \quad \mu^d : CF(\underline{L}_0, \underline{L}_1) \otimes \cdots \otimes CF(\underline{L}_{d-1}, \underline{L}_d) \rightarrow CF(\underline{L}_0, \underline{L}_d)$$

are given by counting quilted $(d+1)$ -gons whose boundary conditions are given by the \underline{L}_i .

Here we need an enlargement of (generalized) Fukaya category of a Stein manifold to include noncompact admissible Lagrangians. For a Stein manifold M we denote by $\text{Fuk}^\#(M)$ the A_∞ category whose objects are allowable proper generalized Lagrangian correspondences between M and a point. Symplectic manifolds involved in these correspondences can be either Stein or compact. Two such correspondences \underline{L}_0 and \underline{L}_1 satisfy the finite intersection property and we define the set of morphisms between them to be $CF(\mathfrak{C}_c(\underline{L}_0^t \# \underline{L}_1))$. Here we choose c so as to include the intersection points of the Lagrangians. The A_∞ structure is given in the same way as for the generalized Fukaya category of compact manifolds. Lemma 3.3.2 insures that the moduli spaces involved are compact.

For an (m, n) tangle T , let $\Phi(T) = (L_{0,1}, L_{1,2}, \dots, L_{n-1,n})$ be as in (40). We obtain an A_∞ functor

$$(52) \quad \Phi_T^\# : \text{Fuk}^\#(\mathcal{Y}_m) \rightarrow \text{Fuk}^\#(\mathcal{Y}_n)$$

which is given at the level of objects by $\Phi_T^\#(\underline{L}) = \underline{L} \# \Phi(T)$ for each $\underline{L} \in \text{Fuk}^\#(\mathcal{Y}_m)$. For details on the functor $\#$ in general see [13]. If K is a link then we obtain a functor

$$(53) \quad \Phi_K^\# : \text{Fuk}^\#(pt) \rightarrow \text{Fuk}^\#(pt).$$

The group valued tangle invariant is defined as follows.

Definition 4.3.1.

$$(54) \quad \mathcal{K}h_{symp}(T) = \bigoplus_{\substack{C \in \mathcal{C}_m \\ C' \in \mathcal{C}_n}} HF(L_C, \Phi(T), L_{C'}).$$

Each summand in the above direct sum is equal to the Floer cohomology of the Lagrangians

$$\begin{aligned} \mathcal{L}_0 &= L_C \times L_0 \times L_{1,2} \times \dots \times L_{2k-1,2k} \\ \mathcal{L}_1 &= L_{0,1} \times L_{2,3} \times \dots \times L_{2k+1} \times L_{C'} \end{aligned}$$

in $\mathcal{Y} = \mathcal{Y}_{n_0}^- \times \mathcal{Y}_{n_1} \times \dots \times \mathcal{Y}_n$. If ψ_i is the plurisubharmonic function on \mathcal{Y}_i then \mathcal{Y} is a Stein manifold with the plurisubharmonic function $\psi = \Sigma \psi_i$.

Lemma 4.3.2. *The Lagrangians \mathcal{L}_i , $i = 0, 1$ are allowable.*

Proof. The Lagrangians are exact since they are simply connected submanifolds of exact manifolds. A point $m = (m_0, \dots, m_{2k+1})$ of tangency of \mathcal{L}_0 to a level set of ψ is a critical point of $\psi|_{\mathcal{L}_0}$. Since ψ is the sum of the plurisubharmonic functions on each \mathcal{Y}_i , m_i is a critical point of $\pi_i^* \psi|_{L_{i,i+1}}$. If $L_{i,i+1}$ is noncompact, it is a vanishing cycle on the diagonal either in $\mathcal{Y}_m \times \mathcal{Y}_{m+1}$ or in $\mathcal{Y}_{m+1} \times \mathcal{Y}_m$. In the first case m_i is a critical point of $\pi_i^* \psi = \psi_i$ on \mathcal{Y}_m . Since \mathcal{Y}_i and ψ_i are algebraic, the critical point set is compact. The second case is similar. \square

Theorem 4.3.3. *For any tangle T , $\mathcal{K}h_{symp}(T)$ is well-defined and is independent of the decomposition of T into elementary tangles.*

Proof. The finite intersection condition holds since the Lagrangian correspondences $L_{i,i+1}$ are proper (cf. 3.3.1). We take the parameter R in the construction of the L_{\cap_i} and L_{\cup_i} so that all the intersection points are included in $M_{\leq R}$. Therefore by Proposition 3.3.4 the above Floer cohomology is well-defined. Note that since our Lagrangians are products of 2-spheres, they have a unique spin structure so the Floer cohomology groups above are modules over \mathbb{Z} . (cf. Theorem “Fs” in [4].) Independence of the decomposition follows from the Functoriality Theorem and theorem 4.2.8. \square

It is clear that if K is a $(0, 0)$ -tangle, i.e. a link, then the above invariant equals the original invariant of Seidel and Smith (2.7.1).

4.4. Some computations. We compute $\mathcal{K}h_{symp}$ for elementary tangles. Set

$$\nu = H^*(S^2)\{-1\}.$$

Remember that \oplus denotes unlinked disjoint union. Let $\sigma_{i;m}^+$ and $\sigma_{i;m}^-$ be as in the Figure 6. If Lagrangians $L, L' \subset X \times \mathbb{C}^3$ are obtained from Lagrangians $K, K' \subset X$ by the relative vanishing cycle construction then we can isotope L and L' to $K \times S^2$ and $K' \times S^2$ inside a compact subset. Therefore we get

$$(55) \quad HF(L', L') = HF(K \times S^2, K' \times S^2) = HF(K, K') \otimes H^*(S^2)$$

For a tangle T it follows from the above formula that

$$(56) \quad \mathcal{H}_{symp}(T \oplus \bigcirc) = \mathcal{H}_{symp}(T) \otimes H^*(S^2)\{-1\} = \mathcal{H}_{symp}(T) \otimes \mathcal{V}.$$

The -1 degree shift here comes from the cup in \bigcirc . More generally it follows from the commutation relations in section 4.2 that for any two tangles $kTl, mT'n$ we have

$$(57) \quad \Phi^\#(T \oplus T') = \Phi^\#(T \oplus id_m) \circ \Phi^\#(id_l \oplus T') = \Phi^\#(id_k \oplus T') \circ \Phi^\#(T \oplus id_n).$$

This gives an injection

$$(58) \quad \mathcal{H}_{symp}(T \oplus T') \hookrightarrow \mathcal{H}_{symp}(T) \otimes \mathcal{H}_{symp}(T')$$

For links L, L' this becomes an isomorphism

$$(59) \quad \mathcal{H}_{symp}(L \oplus L') = \mathcal{H}_{symp}(L) \otimes \mathcal{H}_{symp}(L').$$

Following Khovanov [10] we define

$$(60) \quad H^n = \mathcal{H}_{symp}(id_n) = \bigoplus_{C, C' \in \mathcal{C}_n} HF(L_C^t, L_{C'}).$$

It follows from (55) and 4.2.4 that each summand in the above direct sum equals $H^*(S^2)^{\otimes k}$ where k is the number of circles made by attaching C^t to C' . Therefore (60) equals Khovanov's H^n as an abelian group.

We now give a recursive decomposition of H^m . Denote by c'_m the subset of c_m consisting of elements which contain \cap_1 , i.e. elements which contain an arc between points 1 and 2, and denote by c''_m its complement. (1 can be replaced with any $1 \leq i \leq 2m-1$.) c'_m is in one-to-one correspondence with c_{m-1} . We have a map $c''_m \rightarrow c_{m-1}$ $a \mapsto a'$ given by joining the two strands of a that stem from 1 and 2. Let $cc_m^1 \subset c''_m \times c''_m$ be the subset of all (a, b) such that the arcs passing through points 1 and 2 in a^tb form a single circle. If (a, b) is in the complement of cc_m^1 then the arcs passing through points 1 and 2 in a^tb form two circles.

Let $a, b \in c'_m$. If \bar{a} denotes a with the \cap_1 removed then we have $HF(L_a^t, L_b) = HF(\bar{a}, \bar{b}) \otimes \nu\{1\}$ by (55). This contributes a summand of $H^{m-1} \otimes \nu\{1\}$ to H^m . Set

$$\tilde{H}^m = \bigoplus_{a \in c'_m, b \in c''_m} HF(L_a^t, L_b).$$

The embedded circle C in $a^t b$ which passes through points 1 and 2 contributes a factor of $\nu\{1+i\}$ to $HF(L_a^t, L_b)$ where i is the number of other pairs of points $2k-1, 2k$ which C passes through. This follows from (49). We can set

$$\tilde{H}^m = H^{m'} \otimes \nu\{1\}$$

where $\nu\{1\}$ is the “local” contribution of the circle containing \cap_1 or \cup_1 . This means that if $a \in c'_m, b \in c''_m$ and we modify the strands of $a^t b$ passing through 1 and 2 only in a small neighborhood of the points 1 and 2 then we alter only the second factor in $H^{m'} \otimes \nu\{1\}$. Also denote by \tilde{H}_1^m and \tilde{H}_2^m the contribution of $c c_m^1$ and its complement to H^m . Again we can write $\tilde{H}_1^m = H_1^m \otimes \nu\{1\}$ and $\tilde{H}_2^m = H_2^m \otimes \nu\{1\} \otimes \nu\{1\}$ where $\nu\{1\}$ resp. $\nu\{1\} \otimes \nu\{1\}$ are “local” contributions from the single circle resp. the two circles formed by arcs passing through 1 and 2. Therefore we get

$$(61) \quad H^m = \left((H^{m-1} \oplus H^{m'} \oplus H^{m'} \oplus H_1^m) \otimes \nu\{1\} \right) \bigoplus H_2^m \otimes \nu\{1\} \otimes \nu\{1\}.$$

as abelian groups.

Lemma 4.4.1. *We have*

$$\begin{aligned} \mathcal{K}h_{symp}(\sigma_{i;m}^\pm) &= \left((H^{m-1} \oplus H^{m'} \oplus H^{m'} \oplus H_1^m) \otimes \nu\{1 \mp 2\} \right) \bigoplus H_2^m \otimes \nu\{2\}. \\ \mathcal{K}h_{symp}(\cup_{i;m}) &= (H^{m-1} \otimes \nu) \bigoplus \bigoplus_{a \in c''_m, b \in c_{m-1}} HF(L_{a'}, L_b) \\ \mathcal{K}h_{symp}(\cap_{i;m}) &= (H^{m-1} \otimes \nu\{1\}) \bigoplus \bigoplus_{a \in c''_m, b \in c_{m-1}} HF(L_{a'}, L_b). \end{aligned}$$

Proof. The last two equalities are immediate consequences of (56). The computation for the first equality is similar to that of the rings H^m . The only difference comes from the existence of braids $\sigma_{i;m}^\pm$. As stated before if we modify the strands of a^t passing through 1 and 2 in a small neighborhood of these two points then only the factors $\nu\{1\}$ and $\nu\{1\} \otimes \nu\{1\}$ in (61) change. In the first four direct summands of the decomposition (61) the component of $a^t \sigma_{i;m}^\pm b$ containing the braid contributes the following.

$$HF(L_{\cap_i} L_{\sigma_{i;m}^\pm}, L_{\cup_i}) \{-w(\sigma_{i;m}^\pm)\} = HF(L_{\cap_i} \{\mp 1\}, L_{\cup_i}) \{\mp 1\} = \nu \{1 \mp 2\}$$

The second to last equality is because $\sigma_{i;m}^+ = \sigma_{i;m}$ and $\sigma_{i;m}^- = \sigma_{i;m}^{-1}$ when we ignore the orientation. In the last direct summand the component of $a^t \sigma_{i;m}^\pm b$ containing the braid looks locally like the T_0 in Figure 7. (The figure depicts the case of $\sigma_{i;m}^+$). It is equivalent to $\cap_{1,2} \cap_{3,4} (\sigma_{i;m}^\pm) \cup_{1,2} \cup_{3,4}$. We can use the commutation relation (46) to get $\cap_{1,4} \cap_{2,3} \sigma_{i;m}^{\pm -1} \cup_{1,2} \cup_{3,4}$. Therefore the contribution of T_0 is

$$\begin{aligned} HF(L_{\cap_{1,4}} L_{\cap_{2,3}} L_{\sigma_{i;m}^\pm}^{(-1)}, L_{\cup_{1,2}} L_{\cup_{3,4}}) \{w(\sigma_{i;m}^\pm)\} = \\ HF(L_{\cap_{1,4}} L_{\cap_{2,3}} \{\pm 1\}, L_{\cup_{1,2}} L_{\cup_{3,4}}) \{w(\sigma_{i;m}^\pm)\} = \\ HF(L_{\cap}, L_{\cup}) \{1\} \{\pm 1 - w(\sigma_{i;m}^\pm)\} = \nu \{2\}. \end{aligned}$$

Where we have used commutation relations (45) and (49). □

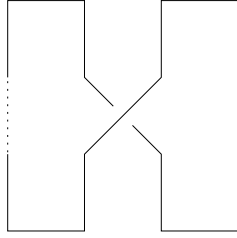


FIGURE 7.

Remark 4.4.2. These values agree with Khovanov invariant (after collapsing the bigrading) as abelian groups. It turns out that H^n has a canonical ring structure for each n and the group $\mathcal{K}h_{\text{sympl}}(T)$, for a (m, n) -tangle T , is a bimodule over the (H^m, H^n) . In a subsequent paper we define a homomorphism

$$(62) \quad \mathcal{K}h_{\text{sympl}}(T') \otimes_{H^j} \mathcal{K}h_{\text{sympl}}(T) \rightarrow \mathcal{K}h_{\text{sympl}}(T' \circ T)$$

for any two tangles iTj and $jT'k$.

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